

THE  
PHYSICAL SOCIETY  
OF  
LONDON.

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PROCEEDINGS.

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VOLUME XXI.—PART V.

OCTOBER 1909.

LONDON:  
TAYLOR AND FRANCIS, RED LION COURT, FLEET STREET.  
1909.

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XXIX. *The Inductance of Two Parallel Wires.* By J. W. NICHOLSON, D.Sc., B.A., Isaac Newton Student in the University of Cambridge\*.

WHEN direct and return currents flow in two wires of great length, and the alternation is not rapid, the effective self-induction  $L$  per unit length of the system may be calculated readily by the method of geometric mean distances† or by simple integration‡. If the wires have radii ( $a$ ,  $b$ ) and permeabilities ( $\mu_1$ ,  $\mu_2$ ), and if  $c$  be the distance between their axes,

$$L = 2 \log \frac{c^2}{ab} + \frac{1}{2}(\mu_1 + \mu_2). \quad . \quad . \quad . \quad (1)$$

But this formula ceases to be of any practical utility in many cases when the frequency of alternation is several thousands per second. Such frequencies are of constant use in practical work. For example, in the measurement of small inductances by Mr. Albert Campbell's method§, it is necessary to employ long leads in order to keep them at some considerable distance from bridge and other circuits. The self-induction of these leads must be small, and a calculation of its value is very desirable. It was therefore suggested to me that I should attack this problem. The general case presents apparently insuperable mathematical difficulty, but the solutions given below appear to include all cases of practical importance. A short statement of these results was given by the author in 'Nature,' Jan. 30th, 1908, but the limitations were not emphasized.

Let the axis of  $z$  be chosen parallel to those of the two wires. Any point in a section defined by a constant value of  $z$  may be conveniently specified by means of polar co-ordinates in two ways. Let these coordinates be  $(r, \theta)$  and  $(\rho, \phi)$ , where  $(r, \rho)$  are the distances of the point from the two axes respectively, and  $(\theta, \phi)$  its orientations measured from a line perpendicular to both axes.

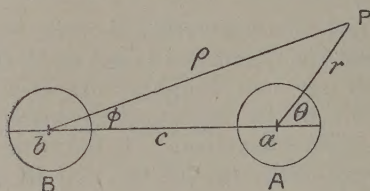
\* Read June 12, 1908.

† Maxwell, 'Electricity and Magnetism.'

‡ Russell, *Alt. Currents*, i. p. 56.

§ Phil. Mag. Jan. 1908.

In the figure A and B are the projections of the axes of the two wires, and  $AB=c$ . A and B will be referred to as the first and second wires respectively. If the first wire were not disturbed by the presence of the second, the electric and



magnetic vectors would be derivable from a vector potential  $H$  parallel to the axis, independent of  $\theta$ , and proportional to a time factor  $e^{ipt}$ , if  $2\pi/p$  be the frequency of alternation.

The electric force in the general case is axial, and will be denoted by  $R$ . There is no axial magnetic component. Let the magnetic components radial and transversal with respect to an origin in the axis of the first wire be  $(a'\beta')$ .

Then in a medium of permeability  $\mu$ ,

$$\alpha' = \frac{1}{\mu r} \frac{\partial H}{\partial \theta}, \quad \beta' = -\frac{1}{\mu} \frac{\partial H}{\partial r} \dots \dots \dots (2)$$

All vectors are independent of  $z$ . The current is parallel to the axis, and of magnitude

$$\begin{aligned} w &= \frac{1}{4\pi r} \left\{ \frac{\partial}{\partial r} (r\beta') - \frac{\partial \alpha'}{\partial \theta} \right\} \\ &= -\frac{1}{4\pi\mu} \left\{ \frac{\partial^2 H}{\partial r^2} + \frac{1}{r} \frac{\partial H}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H}{\partial \theta^2} \right\}. \end{aligned}$$

If the medium have specific inductive capacity  $K$ , and specific resistance  $\sigma$ ,

$$\begin{aligned} w &= \frac{R}{\sigma} + \frac{K}{4\pi} \frac{\partial R}{\partial t} \\ &= \left( \frac{1}{\sigma} + \frac{iKp}{4\pi} \right) R \end{aligned}$$

for the periodic vibration.

Moreover, so far as inductive action is concerned,

$$R = -\frac{\partial H}{\partial t}, \quad \dots \dots \dots (3)$$

and thus,  $H$  being entirely due to inductive agency,

$$\frac{\partial^2 H}{\partial r^2} + \frac{1}{r} \frac{\partial H}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H}{\partial \theta^2} - 4\pi \mu \nu p \left\{ \frac{1}{\sigma} + \frac{\epsilon K p}{4\pi} \right\} H = 0. \quad (4)$$

In the wire, the current may be regarded as due to ohmic conductivity alone, and writing

$$k^2 = -4\pi \mu \nu p / \sigma, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

then

$$\frac{\partial^2 H}{\partial r^2} + \frac{1}{r} \frac{\partial H}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H}{\partial \theta^2} + k^2 H = 0, \quad . \quad . \quad . \quad (6)$$

whose solution finite at the axis of the wire is

$$H = \sum_0^\infty A_n J_n(kr) \cos n\theta, \quad . \quad . \quad . \quad . \quad (7)$$

where

$$J_n(kr) = \frac{k^n r^n}{2^n n!} \left\{ 1 - \frac{k^2 r^2}{2^2 1! n+1} + \frac{k^4 r^4}{2^4 2! n+1 n+2} \dots \right\}, \quad (8)$$

and is the ordinary Bessel function of the first kind. In the surrounding medium,  $\frac{1}{\sigma}$  may be neglected, and writing  $1/K\mu = C^2$ , where  $C$  is the velocity of propagation in the medium,

$$\frac{\partial^2 H}{\partial r^2} + \frac{1}{r} \frac{\partial H}{\partial r} + \frac{1}{r^2} \frac{\partial^2 H}{\partial \theta^2} + h^2 H = 0, \quad . \quad . \quad . \quad (9)$$

where  $h = p/C$ .

The solution proper to the space outside the wire, satisfying the condition of being finite at infinity, is

$$H = \sum_0^\infty B_n K_n(\nu hr) \cos n\theta, \quad . \quad . \quad . \quad (10)$$

where

$$K_n(\nu hr) = \int_0^\infty e^{-\nu hr \cosh \phi} \cosh n\phi d\phi. \quad . \quad . \quad (11)$$

This function is connected with the more usual Bessel function of the second kind, introduced by Hankel, by the formula \*

$$K_n(\nu hr) = -\frac{1}{2} \epsilon^{-n} \{ Y_n(hr) + i\pi J_n(hr) \}, \quad . \quad (12)$$

\* *Vide e. g. Somme, Math. Ann. 16, 1880.*



$h$  being real, and, if  $\gamma = .577 \dots$

$$Y_n(x) \equiv 2J_n(x) \left\{ \log \frac{x}{2} + \gamma \right\} - \left( \frac{2}{x} \right)^n \left\{ n-1! + \frac{n-2!}{1!} \left( \frac{x}{2} \right)^2 \dots \right\} \\ - \left( \frac{x}{2} \right)^n \left\{ \frac{1}{n!} S_n - \frac{1}{1! n+1!} \left( \frac{1}{1} + S_n \right) \left( \frac{x}{2} \right)^2 + \frac{1}{2! n+2!} \left( \frac{1}{1} + \frac{1}{2} + S_n \right) \left( \frac{x}{2} \right)^4 + \dots \right\}, \quad (1)$$

where

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}. \quad \dots \quad (14)$$

### *The Single Wire.*

We consider first the distribution of vectors in and about a single wire conveying an alternating current, whose distribution is undisturbed by the presence of a second wire. All vectors are independent of orientation, and the impressed electric force may be supposed equal at all points of the wire, the resistivity ( $\sigma$ ) of the latter being constant. Let  $Ee^{ipt}$  be the line integral of this force per unit length. Then the total force, inductive and non-inductive, in the wires is

$$R = E - \frac{\partial H}{\partial t} \\ = E - ip A J_0(kr),$$

since  $H$  does not depend on  $\theta$ . The time factor is ignored throughout.

The corresponding force outside is

$$R = - \frac{\partial H}{\partial t} = - ip B K_0(ia),$$

and by continuity, if  $a$  be the radius of the wire,

$$E - ip A J_0(ka) = - ip B K_0(ia).$$

The tangential magnetic force ( $\beta$ ) at the surface is also continuous, and therefore, if  $\mu$  be the permeability of the wire,

$$\frac{1}{\mu} A k J_0'(ka) = ia B k_0'(ia),$$

we deduce that

$$A = \frac{-E}{ip} K_0'(ia) / (K_0'(ia) J_0(ka) - \frac{k}{\mu h} J_0'(ka) K_0(ia)). \quad (15)$$

$$B = \frac{kE}{p h \mu} J_0'(ka) / (K_0'(ia) J_0(ka) - \frac{k}{\mu h} J_0'(ka) K_0(ia)), \quad (16)$$

and the vector potential in the wire is

$$H = A J_0(kr), \quad . . . . . (17)$$

and outside is

$$H = B K_0(\iota hr). \quad . . . . . (18)$$

The current crossing any section is the line integral of magnetic force round the section, divided by  $4\pi$ , and becomes

$$\begin{aligned} \varpi &= \frac{1}{4\pi} 2\pi a(\beta)_{r=a} \\ &= Aka J_0'(ka)/2\mu, \quad . . . . . (19) \end{aligned}$$

and with the above value of  $A$ ,

$$\frac{E}{\varpi ip} = -\frac{2\mu}{k} \left\{ \frac{J_0(ka)}{J_0'(ka)} - \frac{k}{\mu h \iota} \frac{K_0(\iota ha)}{K_0'(\iota ha)} \right\}, \quad . (20)$$

and the effective self-induction is the real part of the expression on the right. The effect due to variability of current in the section, deduced from the first term, is in accord with the result obtained by Lord Kelvin\* and otherwise by Lord Rayleigh†. Maxwell and Heaviside have also given other forms of this solution.

*Transformation of Bessel Functions to a new origin.*

We have by the integral formula for the function finite at infinity,

$$K_n \{ \iota(\overline{c+r}) \} = \int_0^\infty e^{-\iota\overline{c+r} \cosh \phi} \cosh n\phi \, d\phi.$$

But

$$e^{\frac{1}{2}c(z-\frac{1}{z})} = J_0(c) + \sum_1^\infty J_n(c) \left\{ z^n + \frac{(-)^n}{z^n} \right\}, \quad . (21)$$

by an ordinary definition of Bessel functions of the first kind whose order is an integer.

Writing  $z = -\iota e^\phi$ ,

$$e^{-\iota c \cosh \phi} = J_0(c) + 2 \sum_1^\infty (-\iota)^n J_n(c) \cosh n\phi. \quad (22)$$

if the series be convergent.

\* Presidential Address to Inst. of Elect. Engineers, 1889. Math. and Phys. Papers, vol. iii. p. 492.

† *Vide* Phil. Mag. 1886, where a complete practical solution may be found.

Thus

$$K_n(\iota \cdot \overline{c+r})$$

$$\begin{aligned} &= \int_0^\infty d\phi \cdot e^{-\iota r \cosh \phi} \cosh n\phi \cdot \{J_0(c) + 2 \sum_1^\infty (-\iota)^s J_s(c) \cosh s\phi\} \\ &= J_0(c) \int_0^\infty e^{-\iota r \cosh \phi} \cosh n\phi d\phi + \sum_1^\infty (-\iota)^s J_s(c) \int_0^\infty e^{-\iota r \cosh \phi} \left\{ \frac{\cosh n+s\phi}{+\cosh n-s\phi} \right\} d\phi \\ &= J_0(c) K_n(\iota r) + \sum_{s=1}^\infty (-\iota)^s J_s(c) \{K_{n+s}(\iota r) + K_{n-s}(\iota r)\}, \dots \dots \dots (23) \end{aligned}$$

where  $K_{n-s}(\iota r)$  is identical with  $K_{s-n}(\iota r)$ .

This may also be written

$$K_n(\iota \cdot \overline{c+r}) = J_0(r) K_n(\iota c) + \sum_{s=1}^\infty (-\iota)^s J_s(r) \{K_{n+s}(\iota c) + K_{n-s}(\iota c)\} \dots \dots \dots (24)$$

Now the series

$$K_n(\iota hc) J_0(hr) + \sum_{s=1}^\infty (-\iota)^s \{K_{n+s}(\iota hc) + K_{n-s}(\iota hc)\} J_s(hr) \cos s\theta$$

is a solution of the equation (9) which, when  $\theta=0$ , takes the form of the right-hand side of (24). Such a solution is also  $\cos n\phi \cdot K_n(\iota h\rho)$ , where  $\rho$  and  $\phi$  are connected with  $r$  and  $\theta$  as in the figure.

$$\text{Thus} \quad \rho^2 = r^2 + c^2 + 2cr \cos \theta. \quad \dots \dots \dots (25)$$

$$\tan \phi = \frac{r \sin \theta}{c + r \cos \theta}, \quad \dots \dots \dots (26)$$

and when  $\theta=0$ ,  $\cos n\phi K_n(\iota h\rho) = K_n\{\iota h(c+r)\}$ ,

Thus the two solutions only differ by a function vanishing with  $\theta$ , and therefore capable of expression in a series of sines. This function is at once seen to be zero, by an application of Fourier's theorem. The function  $\cos n\phi K_n(\iota h\rho)$  and the series last written do not change sign with  $\theta$ . Thus if  $(\rho, \phi)$  are connected with  $(r, \theta)$  as before,

$$\cos n\phi \cdot K_n(\iota h\rho)$$

$$= K_n(\iota hc) J_0(hr) + \sum_{s=1}^\infty (-\iota)^s \{K_{n+s}(\iota hc) + K_{n-s}(\iota hc)\} J_s(hr) \cos s\theta \quad (27)$$

and similarly

$$\cos n\theta \cdot K_n(\iota hr) = K_n(\iota hc) J_0(h\rho)$$

$$+ \sum_{s=1}^\infty \iota^s \{K_{n+s}(\iota hc) + K_{n-s}(\iota hc)\} J_s(h\rho) \cos s\phi. \quad (28)$$



*Effect of the Return Wire.*

It has been shown that if an impressed force  $E$  (ignoring a time-factor  $e^{ipt}$ ) acts per unit length in a single wire of radius  $a$ , the vector potential produced outside is

$$H = BK_0(\iota hr),$$

where  $h = p/C$ ,  $C$  being the velocity of propagation of electromagnetic disturbances in the medium, and

$$B = \frac{kE}{ph\mu} J_0'(ka) / (K_0'(\iota ha) J_0(k\alpha) - \frac{k}{\mu h \iota} J_0'(k\alpha) K_0(\iota ha)),$$

where the origin is in the axis of the wire, and

$$k^2 = -4\pi\mu p/\sigma.$$

It will be convenient to write

$$\alpha = K_0'(\iota ha) J_0(ka) - \frac{k}{\mu h \iota} J_0'(ka) K_0(\iota ha). \quad (29)$$

where  $\alpha$  depends entirely on the first wire.

The corresponding function of the second wire will be called  $\beta$ , and its radius  $b$ .

Thus

$$B = \frac{kE}{ph\mu\alpha} J_0'(ka). \quad (30)$$

Writing  $n=0$  in (28), it appears that

$$K_0(\iota hr) = K_0(\iota hc) J_0(h\rho) + 2 \sum_{s=1}^{\infty} \iota^s K_s(\iota hc) J_s(h\rho) \cos s\phi, \quad (31)$$

and if  $hb$  is small, as in all applications proposed in this paper, this series is rapidly convergent in the neighbourhood of the second wire.

We may write

$$\theta_0 = K_0(\iota hc), \quad \theta_s = 2\iota^s K_s(\iota hc) \quad s \neq 0, \quad (32)$$

and the vector potential  $H$ , which may be regarded as incident on the second wire, becomes

$$H = B\theta_0 J_0(h\rho) + \sum_1^{\infty} B \cdot \theta_s \cdot J_s(h\rho) \cos s\phi. \quad (33)$$

We neglect, in the present section, the terms dependent on orientation. This will afterwards appear to be equivalent to a neglect of terms of the order  $(a/c)^2$ , so that the wires are

not close together. Thus

$$H = B\theta_0 J_0(h\rho), \quad . \quad . \quad . \quad (34)$$

is incident on the second wire. A secondary disturbance will be thrown off, of the form

$$H = CK_0(\iota h\rho), \quad . \quad . \quad . \quad (35)$$

zero at infinity, and inside the wire, a disturbance

$$H = D J_0(k'\rho), \quad . \quad . \quad . \quad (36)$$

finite at the origin, will be introduced, where

$$k'^2 = -4\pi\nu\rho/\sigma', \quad . \quad . \quad . \quad (37)$$

$\nu$  being the permeability of the second wire.

Since the electric force  $R$  is continuous at the surface,  $H$  is also continuous (for this disturbance due to induction), and therefore

$$DJ_0(k'b) = CK_0(\iota hb) + B\theta_0 J_0(hb),$$

and because the magnetic force is also continuous,

$$\frac{1}{\nu} k' D J_0'(k'b) = \iota h CK_0'(\iota hb) + B h \theta_0 J_0'(hb).$$

Adopting the notation

$$\left. \begin{aligned} \delta_1 &= J_0(ka) J_0'(ha) - \frac{k}{\mu h} J_0(ha) J_0'(ka) \\ \delta_2 &= J_0(k'b) J_0'(hb) - \frac{k'}{\nu h} J_0(hb) J_0'(kb) \end{aligned} \right\}, \quad . \quad (38)$$

it appears that

$$\left. \begin{aligned} C &= \iota B \theta_0 \delta_2 / \beta \\ D &= \iota B \theta_0 / h b \beta \end{aligned} \right\}, \quad . \quad . \quad . \quad (39)$$

where the numerator of  $D$  has been simplified by the reduction

$$\begin{aligned} & \iota K_0'(\iota hb) J_0(hb) - K_0(\iota hb) J_0'(hb) \\ &= \frac{1}{2} (J_0'(hb) Y_0(hb) - J_0(hb) Y_0'(hb)) \\ &= -\frac{1}{hb}, \end{aligned}$$

by the usual property of the Hankel solution  $Y_0$ .

The second wire, therefore, throws out a disturbance of vector potential

$$H = \frac{\iota B \theta_0 \delta_2}{\beta} K_0(\iota h\rho). \quad . \quad . \quad . \quad (40)$$

*Potential independent of Orientation.*

In the calculation just made, the terms of a transformed series of Bessel functions involving orientation were neglected. We now trace the series of compensating disturbances which may be regarded as thrown off by the wires, and form their sum under the same condition. This leads to a very accurate result if the wires are not very close together.

A vector potential  $H=BK_0(\iota hr)$  leaving the first wire has, for its main term near the second, a magnitude  $B\theta_0 J_0(h\rho)$ , and causes (1) an inside potential

$$H=-\frac{B\theta_0}{\iota hb\beta}J_0(k'\rho), \quad . \quad . \quad . \quad . \quad . \quad (41)$$

and (2) a reflexion of vector potential from that wire, of

$$H=\frac{\iota B\theta_0\delta_2}{\beta}K_0(\iota h\rho). \quad . \quad . \quad . \quad . \quad . \quad (42)$$

This is again incident on the first wire, having for a leading term there,

$$H=\frac{\iota B\theta_0\delta_2}{\beta}.\theta_0 J_0(hr), \quad . \quad . \quad . \quad . \quad (43)$$

and causing an inside potential

$$H=\frac{\iota B\theta_0^2\delta_2}{\beta}.\frac{\iota}{ha\alpha}.J_0(hr), \quad . \quad . \quad . \quad . \quad (44)$$

and a reflexion

$$\left. \begin{aligned} H &= \frac{\iota B\theta_0^2\delta_2}{\beta}.\frac{\iota\delta_1}{\alpha}.K_0(\iota hr) \\ &= -B\theta_0^3.\frac{\delta_1\delta_2}{\alpha\beta}.K_0(\iota hr). \end{aligned} \right\} \quad . \quad . \quad . \quad (45)$$

This process may be continued. The successive waves of reflexion form a geometric series of common ratio

$$\theta_0^2\delta_1\delta_2/\alpha\beta.$$

The subsequent reduction indicates that in general, with a frequency of the magnitude dealt with in this paper, this series is convergent. The effect inside the second wire, of a potential

$$H=BK_0(\iota hr),$$



originally supposed to leave the first, is a potential

$$H = \frac{iB\theta_0}{hb\beta} J_0(k'\rho) \left\{ 1 - \theta_0^2 \frac{\delta_1\delta_2}{\alpha\beta} + \theta_0^4 \frac{\delta_1^2\delta_2^2}{\alpha^2\beta^2} - \dots \right\}$$

$$= \frac{i\theta_0}{hb\beta} \frac{B}{1 + \theta_0^2 \frac{\delta_1\delta_2}{\alpha\beta}} J_0(k'\rho). \quad (46)$$

Moreover, the effect on the potential in the first wire, due to the waves caused by the potential originally leaving that wire, is

$$H = - \frac{B\theta_0^2\delta_2}{ha\alpha\beta} \left\{ 1 - \theta_0^2 \frac{\delta_1\delta_2}{\alpha\beta} + \dots \right\} J_0(hr)$$

$$= - \frac{B\theta_0^2\delta_2}{ha\alpha\beta} \frac{J_0(hr)}{1 + \theta_0^2 \frac{\delta_1\delta_2}{\alpha\beta}}. \quad (47)$$

A similar potential  $H = B'K_0(ih\rho)$  leaving the second wire produces secondary potentials

$$H = \frac{i\theta_0}{ha\alpha} \frac{B'}{1 + \theta_0^2 \frac{\delta_1\delta_2}{\alpha\beta}} J_0(kr) \quad (48)$$

$$H = - \frac{B'\theta_0^2\delta_1}{hb\alpha\beta} \frac{J_0(k'r)}{1 + \theta_0^2 \frac{\delta_1\delta_2}{\alpha\beta}}, \quad (49)$$

in the first and second respectively, and as

$$B = \frac{k}{ph\mu} \frac{EJ_0'(ka)}{\alpha} \text{ by (16),}$$

so

$$B' = \frac{k'}{phv} \frac{EJ_0'(k'b)}{\beta}, \quad (50)$$

so that the secondaries may be exactly analogous to the first group.

The value of  $H$  in the first wire is therefore, to this order,

$$H = \left\{ A - \frac{B\theta_0^2\delta_2}{ha\alpha\beta} \frac{1}{1 + \theta_0^2 \frac{\delta_1\delta_2}{\alpha\beta}} + \frac{i\theta_0 B'}{ha\alpha} \frac{1}{1 + \theta_0^2 \frac{\delta_1\delta_2}{\alpha\beta}} \right\} J_0(kr) \quad (51)$$

with a corresponding total current

$$\varpi_1 = \frac{ka}{2\mu} J_0'(ka) \cdot \frac{H}{J_0(kr)}. \quad (52)$$

As previously indicated,  $ha$  and  $hb$  are to be supposed very small, so that  $ha J_0'(ka)$  is of order  $(ha)^2$ .

Neglecting this order of magnitude, a justification for the process appearing later,

$$\left. \begin{aligned} \delta_1 &= -\frac{k}{\mu h} J_0'(ka) \\ \delta_2 &= -\frac{k'}{\nu h} J_0'(k'b) \end{aligned} \right\}, \quad . \quad . \quad . \quad (53)$$

and therefore

$$B = \frac{-E}{p\alpha} \delta_1, \quad B' = \frac{-E}{p\beta} \delta_2; \quad . \quad . \quad . \quad (54)$$

and in the first wire, on reduction,

$$H = \frac{-E J_0(kr)}{p h a \alpha} \left( 1 + \frac{i \theta_0 \delta_2}{\beta} \right) / \left( 1 + \frac{\theta_0^2 \delta_1 \delta_2}{\alpha \beta} \right), \quad . \quad (55)$$

and in the second,

$$H = \frac{-E J_0(k'r)}{p h b \beta} \left( 1 + \frac{i \theta_0 \delta_1}{\alpha} \right) / \left( 1 + \frac{\theta_0^2 \delta_1 \delta_2}{\alpha \beta} \right). \quad . \quad (56)$$

The total currents are

$$\varpi_1 = \frac{E \delta_1}{2p\alpha} \left( 1 + \frac{i \theta_0 \delta_2}{\beta} \right) / \left( 1 + \frac{\theta_0^2 \delta_1 \delta_2}{\alpha \beta} \right), \quad . \quad . \quad . \quad (57)$$

$$\varpi_2 = \frac{E \delta_2}{2p\beta} \left( 1 + \frac{i \theta_0 \delta_1}{\alpha} \right) / \left( 1 + \frac{\theta_0^2 \delta_1 \delta_2}{\alpha \beta} \right), \quad . \quad . \quad (58)$$

by (52).

If  $L$  is the self-induction of the system per unit length,  $iLp$  is the imaginary part of  $\frac{E}{w_1} + \frac{E}{w_2}$ , or of

$$2p \left( 1 + \frac{\theta_0^2 \delta_1 \delta_2}{\alpha \beta} \right) \left\{ \frac{1}{\delta_1 \left( 1 + \frac{i \theta_0 \delta_2}{\beta} \right)} + \frac{1}{\delta_2 \left( 1 + \frac{i \theta_0 \delta_1}{\alpha} \right)} \right\}. \quad (59)$$

In particular, if the wires be equal and of the same material, as will usually occur in practice,  $iL$  is the imaginary part of

$$\frac{4\alpha}{\delta_1} \left( 1 + \frac{\theta_0^2 \delta_1^2}{\alpha^2} \right) / \left( 1 + \frac{i \theta_0 \delta_1}{\alpha} \right), \quad \text{or of} \quad \frac{4\alpha}{\delta_1} \left( 1 - \frac{i \theta_0 \delta_1}{\alpha} \right). \quad (60)$$

*Case of a Small Frequency.*

When  $p$  is small,  $ha = pa/C$  is exceedingly small for a wire not of excessive thickness, and may be safely ignored as in the above calculation, even in its first power, and thus

$$\begin{aligned} K_0(ia) &= -\frac{1}{2} \{ Y_0(ha) + i\pi J_0(ha) \} \\ &= -\left( \gamma + \frac{i\pi}{2} + \log \frac{ha}{2} \right) \\ &= -\log ha \quad \text{practically.} \quad . \quad . \quad . \quad (61) \end{aligned}$$

$$K_0'(ia) = -\frac{1}{iha}, \quad . \quad . \quad . \quad (62)$$

so that on reduction,

$$\alpha = \frac{k}{h\mu} J_0'(ka) \left\{ \log ha - \frac{\mu J_0(ka)}{ka J_0'(ka)} \right\}, \quad . \quad (63)$$

and

$$-\frac{i\delta_1}{\alpha} = 1 \left/ \left\{ \log(ha) - \frac{\mu J_0}{ka J_0'} \right\} \right. . \quad . \quad . \quad (64)$$

$$\theta_0 = K_0(ihc) = -\log hc.$$

If  $hc$  is not too great

$$1 + \frac{i\theta_0\delta_1}{\alpha} = \frac{\log \overline{h^2ac} - \mu J_0/ka J_0'}{\log ha - \mu J_0/ka J_0'},$$

and has the value 2 for a small value of  $h$ .

$$1 + \frac{\theta_0^2 \delta_1 \delta_2}{\alpha \beta} = - \left\{ \log \frac{c^2}{ab} + \epsilon_1 + \epsilon_2 \right\} \log h, \quad . \quad (65)$$

to the same order, where

$$\epsilon_1 = \mu J_0(ka) / ka J_0'(ka), \quad . \quad . \quad . \quad (66)$$

and  $\epsilon_2$  is the same function of  $(k', b)$ ,

$$\frac{\delta_1}{\alpha} = \frac{i}{\log h} = \frac{\delta_2}{\beta},$$

and  $\epsilon_1$  reduces practically, under the present circumstances, to

$$-2\mu / k^2 a^2 \left\{ 1 - \frac{k^2 a^2}{4} \right\},$$

and quoting the value of  $k$ , the real part is  $\frac{1}{2}\mu$ , and the self-induction becomes

$$L = 2 \log \frac{c^2}{ab} + \frac{1}{2} (\mu + \nu), \quad . \quad . \quad . \quad (67)$$



in accord with the results deduced from a distribution assumed uniform.

*Case of two equal wires.*

When the wires are equal,  $\iota L$  is the imaginary part of

$$\frac{4\alpha}{\delta_1} \left( 1 - \iota \theta_0 \frac{\delta_1}{\alpha} \right).$$

When  $h^2 c^2$  and  $h^2 a^2$  are both neglected, this leads to

$$L = 4 \log \frac{c}{a} + \frac{4\mu}{x} \frac{\text{ber } x \text{ber}' x + \text{bei } x \text{bei}' x}{(\text{ber}' x)^2 + (\text{bei}' x)^2}, \quad (68)$$

where, if  $\frac{n}{2\pi}$  is the frequency of alternation per second,

$$x = 2a \sqrt{\frac{\pi \mu n}{\sigma}}. \quad (69)$$

The functions  $\text{ber } x$ ,  $\text{bei } x$ , are those introduced by Lord Kelvin\*, and subsequently tabulated.

To this order of approximation, the result might have been written down at once. For  $c/a$  has already been assumed so great that the distribution may be regarded as made up of cylindrical current sheets. The number of linkages of the lines of force outside the cylinders† is  $4 \log c/a$ , and the linkages due to the lines of force wholly in the metal give rise to the second part of (68)‡.

To the same order the approximation for two wires becomes

$$L = 2 \log \frac{c^2}{ab} + \frac{2\mu}{x} \frac{\text{ber } x \text{ber}' x + \text{bei } x \text{bei}' x}{(\text{ber}' x)^2 + (\text{bei}' x)^2} + \frac{2\nu}{y} \frac{\text{ber } y \text{ber}' y + \text{bei } y \text{bei}' y}{(\text{ber}' y)^2 + (\text{bei}' y)^2} \quad (70)$$

where

$$x = 2a \sqrt{\frac{\pi \mu n}{\sigma}}, \quad y = 2b \sqrt{\frac{\pi \nu n}{\sigma}}, \quad (71)$$

and may also be written down at once.

When the vibration becomes more rapid, even though  $h^2 a^2$  may be neglected in comparison with unity, the latter cannot

\* *Loc. cit. ante.*

† Russell, *Alternating Currents*, i. p. 56 *et seq.*

‡ *Vide* Gray and Matthews, *Treatise on Bessel Functions*.

itself be neglected in comparison with  $\log(ha)$ . Thus (61) requires modification, and it appears that

$$\frac{-\iota\delta_1}{\alpha} = 1 \left/ \left\{ \log \frac{(ha)}{2} + \gamma + \frac{\iota\pi}{2} \right\} - \mu J_0/kaJ_0' \right\},$$

and  $\iota L$  is the imaginary part of

$$\frac{4\alpha}{\delta_1} \left( 1 - \iota\theta_0 \frac{\delta_1}{\alpha} \right)$$

or of

$$-4\iota \left\{ \log \frac{ha}{2} + \gamma + \frac{\iota\pi}{2} - \frac{\mu J_0}{kaJ_0'} \right\} - 4\iota \cdot \left\{ -\log \frac{h\iota}{2} - \gamma - \frac{\iota\pi}{2} \right\}$$

leading to the same value as before.

But when the wires are different, the new approximation thus calculated differs from the old, and is very cumbersome. It is therefore to be desired that, in practical work involving a knowledge of the inductance of parallel leads, the leads should be of the same size and material. The formula for inductance is then of the same accuracy over a wider range of frequency.

Cases may arise in practice in which  $c$  is so great in comparison with  $a$  that the neglect of  $h^2a^2$  may not involve that of  $h^2c^2$ . Tables of the Bessel functions of the second kind must then be employed.  $L$  is the real part of

$$-4 \left\{ \log \frac{ha}{2} + \gamma + \frac{\iota\pi}{2} - \frac{\mu J_0}{kaJ_0'} \right\} - 4K_0(\iota hc),$$

and becomes

$$L = 2Y_0(hc) - 4 \left\{ \log \frac{ha}{2} + \gamma \right\} + \frac{4\mu}{x} \frac{\text{ber } x \text{ ber}' x + \text{bei } x \text{ bei}' x}{(\text{ber}' x)^2 + (\text{bei}' x)^2} \quad (72)$$

where  $x$  has its previous value, and  $Y_0$  represents the function introduced by Hankel. Tables of this function have been given by B. A. Smith\*.

If  $\frac{n}{2\pi}$  is the frequency,

$$h = \frac{n}{3} \cdot 10^{-10} \quad . \quad . \quad . \quad . \quad . \quad (73)$$

A general idea of the range of frequency with which these formulæ may be used, can be obtained at once. The neglected terms are of order  $h^2a^2$  in comparison with those retained.

\* Phil. Mag. (5) xlv. p. 122 (1898).

The radius of a wire is rarely greater than two millimetres, and thus with a frequency of a hundred million per second,  $h^2a^2$  becomes  $1.6 \cdot 10^{-5}$ , which may be neglected at once in comparison with unity. Most frequencies dealt with in practice are therefore within the scope of these formulæ, when the wires are not of large cross-section. But another source of error is present, owing to the neglect of terms depending on orientation. Upon examination, the error is found to be of relative order  $a^2/c^2$ . Even if  $c = 5a$ , the formulæ are sufficiently accurate for most applications. The error is about ten per cent. if  $c = 3a$ , which is the limiting closeness for the majority of practical purposes. More accurate results suited to this case are obtained in the next section.

### *Higher Approximation.*

The terms depending on orientation in the vector potential do not contribute to the total current across any section of a wire, which is proportional to a line integral round the boundary. In considering the supposed waves of potential, it is therefore only necessary to calculate the successive coefficients of the Bessel functions of zero order.

A potential

$$H = BK_0(\imath hr), \dots \dots \dots (74)$$

leaving the first wire, reaches the second as

$$H = B\theta_0 J_0(h\rho) + B\Sigma_1^\infty \theta_s J_s(h\rho) \cos s\phi$$

where

$$\theta_s = 2\imath K_s(\imath hc).$$

If the reflected and internal potentials be respectively

$$\left. \begin{aligned} H &= c_0 K_0(\imath h\rho) + \Sigma_1^\infty c_s K_s(\imath h\rho) \cos s\phi \\ H &= d_\infty J_0(k'\rho) + \Sigma_1 d_s J_s(k'\rho) \cos s\phi \end{aligned} \right\}, \dots (75)$$

the surface conditions readily give

$$c_s = \imath B\theta_s \delta_2 s / \beta_s, \quad d_s = \imath B\theta_s / h\beta_s, \dots \dots (76)$$

where

$$\delta_1^s = J_s(ka)J_s'(ha) - \frac{k}{\mu h} J_s(ha)J_s'(ka), \dots \dots (77)$$

$$\alpha_s = K_s'(\imath ha)J_s(ka) - \frac{k}{\mu h\imath} J_s'(ka)K_s(\imath ha), \dots (78)$$



and  $(\delta_s^s, \beta_s)$  are corresponding functions for the second wire. The potential

$$H = c_0 K_0(\imath h \rho) + \sum_1^\infty c_s K_s(\imath h \rho) \cos s \phi$$

leaves the second wire. But, if the argument of the functions  $K$  be  $\imath hc$  in all cases,

$$K_1(\imath h \rho) \cos \phi = K_1 J_0(hr) + \sum_{s=1}^\infty (-\imath)^s \{K_{s+1} + K_{s-1}\} J_s(hr) \cos s \theta,$$

$$K_2(\imath h \rho) \cos 2\phi = K_2 J_0(hr) + \sum_{s=1}^\infty (-\imath)^s \{K_{s+2} + K_{s-2}\} J_s(hr) \cos s \theta,$$

and so on,

Thus the potential becomes

$$H = A_0 J_0(hr) + \sum_1^\infty A_s J_s(hr) \cos s \theta, \quad . \quad . \quad (79)$$

where

$$A_0 = C_0 K_0 + C_1 K_1 + C_2 K_2 + \dots$$

$$A_1 = -\imath C_0 \cdot 2K_1 - \imath C_1 (K_2 + K_0) - \imath C_2 (K_3 + K_2) \dots$$

$$A_2 = -C_0 \cdot 2K_2 - C_1 (K_3 + K_1) - C_2 (K_4 + K_0) \dots \quad (80)$$

The first term alone contributes to the current crossing any section of the first wire. It causes an internal potential

$$H = -\frac{A_0}{\imath h a \alpha} J_0(kr) \quad . \quad . \quad . \quad (81)$$

in accordance with previous calculation, and a reflected wave

$$H = i \frac{A_0 \delta_1}{\alpha} K_0(\imath hr) + \imath \sum_{s=1}^\infty \frac{A_s \delta_1^s}{\alpha_s} K_s(\imath hr) \cos s \theta. \quad (82)$$

We now introduce the fact that the squares of  $ha$ ,  $hb$ ,  $hc$  may be ignored. Thus

$$K_0 = -\log(hc),$$

$$K_n = -\frac{\imath^{-n}}{2} \{Y_n(hc) + \imath \pi J_n(hc)\}$$

$$= \frac{1}{2} \cdot n-1! \left(\frac{2}{\imath hc}\right)^n \quad n \neq 0,$$

$$J_s(ha) = \left(\frac{ha}{2}\right)^n \cdot \frac{1}{n!},$$

$$K_s'(\imath ha) = -2^{n-1} n! / (\imath ha)^{n+1}.$$

If

$$\lambda_s = \left(1 - \frac{\mu_s J_s(ka)}{ka J'_s(ka)}\right) / \left(1 + \frac{\mu_s J_s(ka)}{ka J'_s(ka)}\right), \quad \dots \quad (83)$$

and  $\mu_s$  be the corresponding quantity for the second wire, it appears on reduction that

$$\begin{aligned} C_s &= -\frac{B\theta_s\mu_s(hb)^{2s}\iota^s}{2^{2s-1}s!s-1!} \\ &= -\frac{B\mu_s h^s b^{2s}\iota^s}{s!2^{s-1}c^s} \cdot \dots \dots \dots (84) \end{aligned}$$

The coefficients of type  $c$  therefore rapidly become small, especially if  $c/b$  is large. If  $hc$  is large and  $hb$  small the convergence is still more rapid, but the first form of  $c_s$  is to be used.  $\theta_s$  itself decreases rapidly as  $hc$  increases.

Accordingly, when  $c$  is so large that the approximate values above cannot be used for the functions  $K_n(\iota hc)$ , the result will be correct to the extent of an error of order  $h^2a^2$ , and this case needs no further consideration.

When  $hc$  is small, on reduction

$$\left. \begin{aligned} A_0 &= C_0 K_0 - B \sum_1^\infty \frac{\mu_s}{s} \left(\frac{b}{c}\right)^{2s}, \\ A_1 &= -\frac{2C_0}{hc} + \frac{2B}{hc} \sum_1^\infty \mu_s \left(\frac{b}{c}\right)^{2s}, \\ A_2 &= \frac{4C_0}{h^2c^2} + \frac{4B}{h^2c^2} \sum_1^\infty (s+1)\mu_s \left(\frac{b}{c}\right)^{2s} \end{aligned} \right\}, \quad \dots \quad (85)$$

and so on, very convergent series being obtained if  $\frac{b}{c}$  is small.

The reflected wave of (82) becomes, near the second wire,

$$H = B_0 J_0(h\rho) + \sum_1^\infty B_s J_s(h\rho) \cos s\phi, \quad \dots \quad (86)$$

where

$$\begin{aligned} B_0 &= \frac{\iota A_0 \delta_1}{\alpha} K_0 + \iota \sum_1^\infty \frac{A_s \delta_1^s}{\alpha_s} K_s, \\ B_1 &= 2\iota K_1 \cdot \frac{\iota A_0 \delta_1}{\alpha} - \sum_1^\infty A_s (K_{s+1} + K_{s-1}). \end{aligned}$$

The ratios  $a/c$ ,  $b/c$ , will be retained to the second power only. To this order,

$$B_0 = \frac{\iota \delta_1 K_0}{\alpha} \left\{ C_0 K_0 - B \mu_1 \frac{b^2}{c^2} \right\} + \lambda_1 C_0 \frac{\alpha^2}{c^2} \quad \dots \quad (87)$$

$B_1$  will only subsequently be required to the first order, and becomes

$$B_1 = \frac{i\delta_1}{\alpha} 2\epsilon K_1 \cdot C_0 K_0 \cdot \dots \dots \dots (88)$$

We note that the change in  $B_0$  due to taking account of  $c_1 c_2 \dots$  is of order  $a^2/c^2$  as stated in the previous section.

So far as the current is concerned, the internal potential produced in the second wire is

$$H = -\frac{B_0}{i\hbar\beta} J_0(k\rho), \quad \dots \dots \dots (89)$$

and a potential

$$H = c_0 J_0(hr) + \sum_1^\infty c_s J_s(hr) \cos s\theta \quad \dots \dots (90)$$

is incident again on the first wire, where

$$\begin{aligned} c_0 &= \frac{i\delta_2}{B} \cdot B_0 K_0 + i\sum_1^\infty B_s \frac{\delta_2^s}{\beta_s} K_s \\ &= \frac{i\delta_2}{B} \cdot B_0 K_0 + \frac{i\delta_2'}{\beta_1} B_1 \quad \text{to order} \quad \frac{a^2}{c^2} \\ &= \rho_1 \rho_2 \left( C_0 K_0 - B\mu_1 \frac{b^2}{c^2} \right) + \lambda_1 \rho_2 C_0 \frac{a^2}{c^2} - \mu_1 \rho_1 C_0 \frac{b^2}{c^2} \end{aligned} \quad (91)$$

where

$$\rho_1 = \frac{i\delta_1 K_0}{\alpha} \quad \rho_2 = \frac{i\delta_2 K_0}{\beta} \quad \dots \dots (92)$$

and

$$c_1 = -2\epsilon K_1 \rho_1 \rho_2 C_0 \cdot \dots \dots \dots (93)$$

If the next wave meets the second wire in the form

$$H = D_0 J_0(h\rho) + \sum_1^\infty D_s J_s(h\rho) \cos s\phi, \quad \dots \dots (94)$$

then in a similar manner,

$$\begin{aligned} D_0 &= -\rho_1 C_0 - \lambda_1 \frac{a^2}{c^2} \rho_1 \rho_2 (-c_0) \\ &= \rho_1^2 \rho_2 \left( C_0 K_0 - B\mu_1 \frac{b^2}{c^2} \right) + 2\lambda_1 \rho_1 \rho_2 C_0 \frac{a^2}{c^2} - \rho_1^2 \mu_1 C_0 \frac{b^2}{c^2} \\ D_1 &= 2\epsilon K_1 \rho_1^2 \rho_2 c_0 \cdot \dots \dots \dots (95) \end{aligned}$$

The leading coefficient in the potential next meeting the first wire is

$$E_0 = \rho_1^2 \rho_2^2 \left( C_0 K_0 - B\mu_1 \frac{b^2}{c^2} \right) + 2\lambda_1 \rho_1 \rho_2^2 C_0 - 2\rho_1^2 \rho_2 \mu_1 C_0 \frac{b^2}{c^2}, \quad (96)$$

and so on, the mode of formation being now obvious.



The potentials containing the coefficients  $A_0, C_0, E_0, G_0 \dots$  contribute to the current in the first wire, and those with  $d_0, B_0, D_0, \dots$  to that in the second.

Now

$$\begin{aligned} A_0 + c_0 + E_0 + \dots &= \left( C_0 K_0 - B \mu_1 \frac{b^2}{c^2} \right) (1 + \rho_1 \rho_2 + \rho_1^2 \rho_2^2 + \dots) \\ &+ \left( \lambda_1 \rho_2 \frac{a^2}{c^2} - \mu_1 \rho_1 \frac{b^2}{c^2} \right) C_0 (1 + 2\rho_1 \rho_2 + 3\rho_1^2 \rho_2^2 + \dots) \\ &= \left( \rho_2 B K_0 - B \mu_1 \frac{b^2}{c^2} \right) / (1 - \rho_1 \rho_2) \\ &+ \rho_2 B \left\{ \lambda_1 \rho_2 \frac{a^2}{c^2} - \mu_1 \rho_1 \frac{b^2}{c^2} \right\} / (1 - \rho_1 \rho_2)^2, \quad (97) \end{aligned}$$

since

$$C = \iota B \theta_0 \frac{\delta_2}{\beta} = \rho_2 B,$$

and

$$\begin{aligned} d_0 + B_0 + D_0 + F_0 + \dots &= \left( B K_0 - \rho_1 B \mu_1 \frac{b^2}{c^2} + \lambda_1 \frac{a^2}{c^2} B \rho_2 \right) / (1 - \rho_1 \rho_2) \\ &+ B \rho_1 \rho_2 \left( \lambda_1 \rho_2 \frac{a^2}{c^2} - \mu_1 \rho_1 \frac{b^2}{c^2} \right) / (1 - \rho_1 \rho_2)^2 \quad (98) \end{aligned}$$

on summation.

Similarly, due to  $B'K_0(\iota l \rho)$  originally leaving the second wire, we have in the first an addition to  $A_0 + C_0 + \dots$  of magnitude

$$\begin{aligned} &\left( B'K_0 - \rho_2 B' \lambda_1 \frac{a^2}{c^2} + \mu_1 \frac{b^2}{c^2} B' \rho_2 \right) / (1 - \rho_1 \rho_2) \\ &+ B' \rho_1 \rho_2 \left( \mu_1 \rho_1 \frac{b^2}{c^2} - \lambda_1 \rho_2 \frac{a^2}{c^2} \right) / (1 - \rho_1 \rho_2)^2. \quad (99) \end{aligned}$$

But

$$B = -\frac{E \delta_1}{p \alpha} = \frac{\iota E \rho_1}{p}, \quad B' = -\frac{E \delta_2}{p \beta} = \iota \frac{E \rho_2}{p} \text{ by (54).}$$

Adding the potential  $H = A J_0(kr)$  obtained in a previous section, and reducing, the total potential in the first wire, which contributes to the current, becomes

$$H = \frac{-E H_1}{p h a z} J_0(kr), \quad (100)$$

where

$$\begin{aligned} H_1 = 1 + & \left( \rho_2 K_0 - \mu_1 \frac{b^2}{c^2} \right) \frac{\rho_1}{K_0} \cdot \left( 1 - \rho_1 \rho_2 \right) \\ & + \frac{\rho_1 \rho_2}{K_0} \left( \lambda_1 \rho_2 \frac{a^2}{c^2} - \mu_1 \rho_1 \frac{b^2}{c^2} \right) \left( 1 - \rho_1 \rho_2 \right)^2 \\ & + \frac{\rho_2}{K_0} \left( K_0 - \rho_2 \lambda_1 \frac{a^2}{c^2} + \mu_1 \rho_2 \frac{b^2}{c^2} \right) \left( 1 - \rho_1 \rho_2 \right) \\ & + \frac{\rho_2^2}{K_0} \left( \mu_1 \rho_1 \frac{b^2}{c^2} - \lambda_1 \rho_2 \frac{a^2}{c^2} \right) \left( 1 - \rho_1 \rho_2 \right)^2, \quad \dots \quad (101) \end{aligned}$$

and the current is

$$\varpi_1 = \frac{E \delta_1}{2p\alpha} \cdot H_1. \quad \dots \quad (102)$$

A similar expression may be written down at once for the current in the second wire, and a formula for the self-induction obtained. When the wires are similar in all respects,

$$\begin{aligned} H &= 1 + \left( \rho_1 K_0 - \lambda_1 \frac{a^2}{c^2} \right) \frac{\rho_1}{K_0} \left( 1 - \rho_1^2 \right) \\ &\quad + \frac{\rho_1}{K_0} \left( K_0 - \rho_1 \lambda_1 \frac{a^2}{c^2} + \rho_2 \lambda_1 \frac{a^2}{c^2} \right) \left( 1 - \rho_1^2 \right) \\ &= \frac{1 + \rho_1}{1 - \rho_1^2} - \frac{\lambda_1}{K_0} \cdot \frac{a^2}{c^2} \cdot \frac{\rho_1}{1 - \rho_1^2}. \end{aligned}$$

If  $L$  be the self-induction of the system,  $iLp$  is the imaginary part of

$$\frac{4p\alpha}{\delta_1} \left\{ \frac{1}{1 - \rho_1} - \frac{\lambda_1 a^2}{K_0 c^2} \cdot \frac{\rho_1}{1 - \rho_1^2} \right\}^{-1},$$

or of

$$\frac{4p\alpha}{\delta_1} \cdot (1 - \rho_1) \cdot \left\{ 1 + \frac{\lambda_1 a^2}{K_0 c^2} \cdot \frac{\rho_1}{1 + \rho_1} \right\} \quad \dots \quad (103)$$

to the same order.

The first term of this expression leads to the results already given. The second term, writing

$$\rho_1 = -K_0 \left/ \left( \log ha - \frac{\mu J_0}{k \omega J_0'} \right) \right.,$$

$$1 + \rho_1 = 2 \text{ practically,}$$

becomes

$$4ip \frac{a^2}{c^2} \lambda_1 \cdot \frac{\log\left(\frac{a}{c}\right) - \frac{\mu J_0}{kaJ_0'}}{\log(h^2ac) - \frac{\mu J_0}{kaJ_0'}}.$$

If R be the real part of

$$4 \frac{a^2}{c^2} \lambda_1 \cdot \frac{\log\frac{a}{c} - \frac{\mu J_0}{kaJ_0'}}{\log h^2ac - \frac{\mu J_0}{kaJ_0'}} \quad \dots \quad (104)$$

where

$$\lambda_1 = \left(1 - \frac{\mu}{ka} \frac{J_1}{J_1'}\right) / \left(1 + \frac{\mu}{ka} \cdot \frac{J_1}{J_1'}\right), \quad \dots \quad (105)$$

and the argument of the Bessel functions is in all cases

$$ka = x\iota^{3/2} = 2a\sqrt{\frac{\pi\mu n}{\sigma}} \cdot \iota^{3/2} \text{ as before.} \quad \dots \quad (106)$$

Then the self-induction is

$$L = 4 \log \frac{c}{a} + \frac{4\mu}{x} \cdot \frac{\text{ber } x \text{ ber}' x + \text{bei } x \text{ bei}' x}{(\text{ber}' x)^2 + (\text{bei}' x)^2} + R. \quad (107)$$

This formula may be used even when  $c = 3a$ , to obtain a result not in error by more than one per cent. For the application of this result, the functions analogous to  $\text{ber } x$ , and  $\text{bei } x$  derived from the Bessel function of order 1, require tabulation.

When the resistivity of the wires is not very great, and the frequency is large, the Bessel functions in these results may usually be replaced by their ordinary asymptotic expansions for large argument, and tables cease to be necessary. The solutions then approximate to that for an infinite frequency, which is well known\*. The formula (72), applicable when the wires are very far apart in comparison with their radii, is chiefly useful in determining a differential effect, for the ends of the wires will have a great influence in most cases to which the formula might be applied if the wires were really infinite.

Trinity College, Cambridge.

\* Russell, *l. c. ante*.

## DISCUSSION.

Mr A. CAMPBELL congratulated the Author and said it was often desirable when measuring self-inductances to check the results by calculation. The Author's formulæ would be useful in finding the inductance of parallel leads, although this was best obtained experimentally by a separate experiment.

Dr RUSSELL said that it required considerable mathematical skill and great patience to attack this problem, and so he thanked the Author for having obtained a solution sufficiently accurate for practical purposes. He pointed out that the formula would have to be used with caution at very high frequencies as the capacity current was then appreciable. He suggested to the Author that he should compute the kinetic energy of the ions or electrified particles by the motion of which modern theory explains the phenomena of the electric current. It has hitherto been the custom to assume that the self-inductance of a current flowing in a circuit can be calculated by finding the magnetic energy only, and this is what Dr Nicholson has done. If we accept modern theory, however, we must admit that part of the total energy of a current is due to the kinetic energy of the ions. To take this into account it is necessary to add the term  $\Sigma mv^2/i^2$  to Dr Nicholson's solutions, where  $m$  is the mass of an ion,  $v$  the mean velocity given to it by the electromotive force, and  $i$  the current. In the case of metallic conduction this term is in general negligible compared with those found by the Author. In cases, however, where the current takes place through gases—as in vacuum tube lighting—the value of the correcting term is appreciable, owing to the high velocity of the electrified particles. In a paper to the Röntgen Society on the measurement of a current through an X-ray tube, Mr Duddell recently described some very interesting phenomena which he had noticed during his difficult but very successful experiments. The speaker thought that probably some of these effects might be explained by taking into account the kinetic energy of the moving ions.

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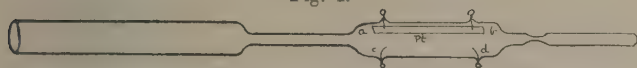
XXX. *A Note on the Photoelectric Properties of Potassium-Sodium Alloy.* By J. A. FLEMING, M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London\*.

IN connexion with further researches on my oscillation valve or glow-lamp radiotelegraphic wave-detector, I was led to examine instances of electronic emission other than that due to the incandescence of metals or carbon *in vacuo*, to ascertain how far rectifying effects with high frequency currents could be obtained by them†. It is well known that under the action of ordinary and ultra-violet light the electropositive metals lose a negative charge of electricity, and it was shown by Elster and Geitel that this photoelectric effect is most pronounced in the case of rubidium, potassium, and the liquid alloy of potassium and sodium.

With the object of examining this effect, experiments were made with various forms of apparatus, and, as the outcome of these, it was found that a convenient mode of preparing a suitable specimen of highly photoelectric metal was as follows:—

A tube of lead glass about 50 cms. long, and 1.5 to 2 cms. internal diameter has a constriction made at one place, and on

Fig. 1.



one side of this a couple of platinum wires are sealed through the glass which are welded to a slip of platinum foil *a, b*, about 5 cms. long and 1 cm. wide (see fig. 1). This foil is fixed close to the inside surface of the tube and opposite to it two other platinum wires *c d* are sealed through the tube.

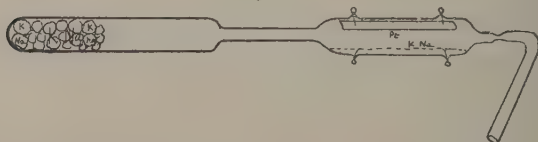
\* Read November 13th, 1908.

† See J. A. Fleming, "On the Conversion of Electric Oscillations into Continuous Currents by means of a Vacuum Valve," *Proc. Roy. Soc. Lond.* vol. lxxiv. p. 476 (1905). Also "The Construction and Use of Oscillation Valves for rectifying High Frequency Currents," *Phil. Mag.* May 1906, p. 659.

The end of this part of the tube is then drawn off and sealed to a length of smaller tube with a constriction in it by which the tube can ultimately be closed. A quantity of metallic potassium and sodium is then cut up into small cubes under naphtha, so as to give a number of clean morsels of the metal.

The operation of filling the glass tube is then as follows :— The fine quill tube at one end is first bent over at an angle of about  $75^\circ$  (see fig. 2), and by means of an indiarubber tube, coal-gas, which has been dried by bubbling through strong sulphuric acid, is led into it. The glass tube as a whole must have previously been well cleansed and carefully dried. When all the air is expelled, the fragments of sodium and potassium are dropped in at the wide open end of the tube, and about equal volumes of the two metals introduced. The open end is then loosely stopped with a plug of asbestos, and the glass tube heated in the blowpipe, and the open end so plugged is melted and sealed. If this is done skilfully, the tube can be sealed off with a rounded end by utilizing the gas-pressure to blow out the softened glass.

Fig. 2.



When cool, the lumps of metallic potassium and sodium are tilted over and should fill up about 10 to 15 cms. of the length of the so sealed off tube (see fig. 2). The tube is then attached to a mercury pump for the purpose of making a high vacuum in it, and should be supported in a slanting position on a sheet of asbestos cardboard and covered by an iron trough. The tube should then be exhausted, and at the same time the metallic potassium and sodium heated so as to melt the lumps into a mass of liquid alloy which, however, will be covered with crusts of oxide.

When a good vacuum has been produced, the whole tube being hot and dry, the constriction in the quill tube may be

melted with the blowpipe and the glass tube sealed off from the pump. If this is properly done, the part of the tube having in it the platinum plate will remain quite clean, and the molten metal or liquid alloy will be contained in the other portion of the tube. It is then easy to tilt the tube and transfer the clean mercury-like alloy of K and Na into the part of the tube containing the platinum plate, so as to make a pool of liquid alloy on the bottom of the tube having an electrical connexion with the outside by means of the platinum wires sealed through the tube, and having over it, and not far from its surface, a platinum plate also in connexion with the outside by sealed-in wires. In so doing, care should be taken that the liquid alloy is not splashed upon the platinum plate, but that the latter is kept quite clean and free from adherent drops of potassium-sodium.

It is better not to attempt to seal off that part of the tube in which the alloy is melted from that part to which it is decanted over, because the glass nearly always cracks and spoils the apparatus in so doing if there is the slightest particle of alloy smeared on it at the part heated in the flame.

A tube so prepared should be carefully handled, as if it is broken the liquid K and Na alloy is spontaneously inflammable and deflagrates violently on coming in contact with moisture. It is convenient to mount it on a wooden stand, which should be placed in an iron tray in case of an accidental breakage. By the aid of such a specimen of photoelectric alloy we can show many interesting experiments. If the tube is supported in a horizontal position, an electric arc contained in a projection lantern, equipped with the ordinary condenser-lens, can be so placed and tilted downwards as to converge on to the brilliant mercury-like surface of the pool of alloy a very concentrated beam of light. If, then, we connect the platinum plate and the alloy by means of wires with a sensitive mirror galvanometer, we find that the impact of the light upon the surface of the alloy not merely facilitates the escape of negative electricity from it, but actually creates an electromotive force and current in the galvanometer circuit, which is in such a direction as to indicate that negative electricity is caused to move, by the

action of the light, through the vacuum from the surface of the alloy to the platinum plate\*.

It has been pointed out by Professor Sir J. J. Thomson in his book 'The Discharge of Electricity through Gases' (see p. 73) that air traversed by ultra-violet light behaves like an electrolyte ; and it is also stated that Stoletow has shown by direct experiment that two different metals immersed in such illuminated air are brought down to the same potential just as when they are immersed in a liquid electrolyte. Again, it is said that when plates of two different metals are opposed to each other and ultra-violet light allowed to fall on one of them, whilst the terminals of a battery of electromotive force  $E$  are connected to them so as to force a current across the space between them, the total E.M.F. in the circuit is not  $E$  but is  $E + M/M'$ , where  $M/M'$  is the contact difference of potential between the metals. There is, however, here an unexplained discrepancy. If metallic potassium or sodium were placed with a platinum plate in a liquid aqueous electrolyte, and the two metals connected by a wire outside so as to construct a voltaic cell, the current through the cell, that is the movement of positive electricity, would be from the potassium or sodium to the platinum through the electrolyte, and therefore the movements of negative electricity would be in the opposite direction. In the case of the photoelectric cell, the movement of negative electricity in the cell is from the K-Na alloy to the platinum. Hence the physical operation of the photoelectric cell is not identical with that of an ordinary voltaic cell. Stoletow has described (*Physikalische Revue*, i. p. 765, 1892) experiments in which two plates of different metals, one perforated with holes, were placed parallel and at a little distance from each other, and ultra-violet light allowed to fall through the openings of the perforated plate on to the other plate. The plates, when connected together, were found to produce a current of electricity in the connecting circuit. Prof. Sir J. J. Thomson,

\* In the case of one tube of alloy prepared as above, my assistant, Mr. G. B. Dyke, noticed that the galvanometer deflexion was largely increased for a time by tilting the tube so as to make momentary contact between the alloy and the platinum, but the increased effect is not permanent.



however, remarks (*loc. cit.* p. 73) that for this to happen in accordance with voltaic principles, the perforated plate must be made of the more electropositive metal, for then only would the negative electricity move towards it across the interspace between the plates. As a matter of fact, the best photoelectric metals are the most electropositive metals. Accordingly, if we imagine a sheet of potassium opposed to a sheet of zinc, then to make the photoelectric current agree in direction with the volta-electric current, we should have to illuminate the zinc plate with ultraviolet light, but keep the potassium plate in the dark. In the present case, the photoelectric current is in the opposite direction to the volta-electric current, assuming the ionized gas replaced by an aqueous electrolyte.

It seems therefore that in the case considered there are two separate sources of electromotive force, viz.: the volta contact-difference of potential of the metals in the cell and a photoelectromotive force due to the illumination; and to these may be added an external electromotive force due to any battery inserted in the circuit. In two tubes, made as above, I found that the electromotive force produced by the incident light was equal to 0.45 volt in one, and in the other tube to 0.6 volt. This E.M.F. was measured by the counter E.M.F. (produced by a shunted cell) which had to be introduced into the photoelectric cell circuit to reduce the galvanometer deflexion to zero. These differences are no doubt due to small differences in composition of the alloy in the two cases, to differences in the intensity of the light used at the time of making the measurements, or to differences in the pressure of the residual gas in the tube. The current produced by the tube of E.M.F. equal to 0.6 volt through a galvanometer of resistance 180 ohms, was found to be 5.4 microamperes. This indicates that the equivalent resistance of the tube is 74,000 ohms on the assumption that the effective E.M.F. is the same when the circuit is open and closed. The current increases very rapidly at first with the intensity of the incident light, but the author has not yet been able to find time to make measurements of the relation between the illumination per square cm. of the surface and the E.M.F. created in the circuit.

By the use of two such cells placed in series, I have found that the separate photoelectromotive forces are additive, and that if such photoelectric cells are joined in series like voltaic cells and separately illuminated, the individual E.M.F.'s are added together in an external circuit connecting the first and last plate. In the case of the two tubes mentioned above, giving separately 0.45 and 0.6 volt E.M.F., the E.M.F. when in series was found to be 1.0 volt.

At one time I contemplated constructing a photoelectric battery of such cells which should give a high E.M.F. by the mere impact of light upon its most electropositive elements.

With the above described cells we can also verify easily a number of interesting observations made by previous experimentalists.

As regards the nature of the radiation which is chiefly effective, we find that for potassium alone, or for the potassium-sodium alloy, the effective rays are the most refrangible ones of the visible spectrum. The ultra-violet light in the arc is considerably filtered out, if not altogether stopped, by the thick glass condenser-lens, and by the glass walls of the tube. We can, however, effect a further separation of visible rays by screens of coloured glass, or solutions or stained gelatine films.

If we interpose in the path of the incident light a very thin film of gelatine stained with a yellow dye or a sheet of ordinary yellow glass, the deflexion of the galvanometer drops almost to zero. The same reduction is effected by a sheet of ruby glass or gelatine film stained red. A green glass cuts off a good deal of the deflexion, but a sheet of cobalt glass reduced it only to about two-thirds, showing that the blue glass is fairly transparent to those rays which can produce this electromotive force.

The galvanometer deflexions increase very rapidly with the intensity of the incident light, and unless an extremely sensitive galvanometer is used it is necessary to throw a very intense beam of light upon the surface of the metal to obtain any marked evidence of the production of electromotive force by the mere action of light upon the surface of the alloy.

If the energy required to produce this photoelectric current comes from the incident light, then the latter must in some degree be absorbed, and the active rays must therefore be those which are absorbed by the photoelectric metal or conversely emitted by it when it is heated. We know well that the light due to the violet line in the flame spectrum of potassium salts passes easily through cobalt glass. Hence the fact that light which has passed through cobalt glass is still exceedingly active in producing the photoelectric effect with potassium, may indicate that it is this ray which is absorbed in its production. In the case of rubidium, Elster and Geitel showed that the yellow and orange rays were relatively more active in the production of the photoelectric effect, and a glance at the flame spectrum of rubidium salts shows that the lines are most numerous in the red, yellow, and green region.

Also, we can confirm easily another observation of the effect of the plane of polarization of the incident light. If we polarize the incident light by a Nicol's prism, and connect the metals of the photoelectric cell with a galvanometer either with or without a battery of 1 or 2 secondary cells inserted in the circuit, then we find at once that the galvanometer deflexion is much greater when the plane of polarization is at right angles to the plane of incidence, than when these planes coincide.

On the electromagnetic theory of light, this may be stated by saying that the effect is a maximum when the electric vector of the plane-polarized light is normal to the surface of the photoelectric metal, and a minimum when it is parallel to it. The theory that fits in best with the above facts, is that the light facilitates or causes an escape of negative corpuscles or electrons from the surface of the photoelectric metal, and to do this it must of course impart to them energy sufficient to give them a velocity enough to carry them beyond the range of attraction of the positive charge which remains behind on the metal.

Since the electropositive metals are those which most easily lose electrons from their atoms, this is in accordance with the observed fact that the most electropositive metals

are the most highly photoelectric substances. Again, if there is an emission under the action of light of electrons from the surface, we have an explanation of another fact easily proved with these tubes, viz. that the formation of a magnetic field parallel to the surface of the alloy greatly reduces the photoelectric current. We can show this easily by the great reduction which occurs in the galvanometer deflexion when even an ordinary horseshoe magnet is placed with its poles across the tube. Again, we can explain on this hypothesis the unilateral conductivity of the vacuous space over the illuminated alloy surface. If we connect the negative pole of a secondary cell to the external terminal of the alloy and the positive to the platinum plate, and insert a galvanometer as usual in the circuit, we greatly increase the galvanometer deflexion, which occurs when no cell is inserted. On the other hand, if the cell is reversed, then, provided the platinum is quite free from splashes of alloy, we cannot reverse the deflexion of the galvanometer.

It is obvious that if a magnetic field is created parallel to the surface of the alloy, a force will be exerted on the negative ion as it moves normally away from the surface, deflecting it from its path, and this will reduce the number of ions which in any time reach the platinum plate, and therefore will diminish the current. Again, the negative electrification of the photoelectric metal will increase the force of propulsion on the ion, and hence increase the emission per unit of time, and therefore increase the observed current.

It is obvious, then, that the ionized gas over the illuminated alloy has a unilateral conductivity, and will conduct the current from a voltaic cell in one direction, but not in the opposite. Negative electricity can be conveyed from the alloy to the platinum across the rarefied air-space, but not in the opposite direction. Hence, an alternating current whether of low or of high frequency, can be rectified, and by interposing such a light-cell in the circuit of a galvanometer, in which circuit high-frequency oscillations are also created by the inductive action of a discharging condenser, I have been able to rectify these oscillations. The action, however, proved to be much more feeble and irregular than the similar



rectifying effect which can be produced by a glow-lamp, and as at that time I had succeeded in finding a far more efficient glow-lamp rectifier or oscillation detector, the experiment with the rectifying effects of potassium-sodium alloy were not continued.

The interesting question, however, still remains as to the source and nature of this photoelectromotive force which is produced by the absorption of light by the surface of highly electropositive metals.

It is much affected by temperature, being increased by heating the alloy and by exposing it to powerful radiation for some time. If a galvanometer is used as described to detect the photoelectric effect, the current is found to be dependent upon the pressure and nature of the residual gas in the tube.

This was confirmed as follows:—A tube was prepared as above described with potassium-sodium alloy. Before sealing it off the pump, dry hydrogen gas was admitted and pumped out and then admitted again, and the tube exhausted to a pressure of about 0.01 mm. This tube was compared with others, in which the residual gas was air, but the exhaustion carried below 0.001 mm. The former tube exhibited only the very smallest evidence of photoelectric effect as measured by the galvanometer current, whereas the highly vacuous tube with residual air under very small pressure exhibited the effects well. The case of other tubes made as described, in which water vapour was accidentally present, which in a short time liberated hydrogen under the action of the metal; these after an interval ceased to give any current, even under the action of the strongest illumination, when the platinum plate was connected with the potassium-sodium alloy through a galvanometer. These experiments and others show that to obtain the effect well a very high vacuum is necessary. In the process of preparing the tube the potassium and sodium should be melted, and the alloy and tube well heated for some time whilst the pump is going to drive off all traces of water vapour, also of hydrocarbon derived from the naphtha in which the metallic potassium is commonly preserved, and especially to drive off the hydrogen which

seems to be occluded in considerable quantity by the alkali-line metals, no doubt derived from the decomposition of water vapour.

The effect of variation of pressure of the gas upon the photoelectric effect was first investigated by Stoletow, and an account of his researches is given in Prof. Sir J. J. Thomson's treatise on the "Conduction of Electricity through Gases" (p. 224). He found that the current increased rapidly as the pressure diminished, which continued until the current reached a maximum value, after which it began to decline, but had a finite value at the lowest attainable pressures.

Stoletow's experiments appear to have been made with zinc plates and ultra-violet light. Owing to the greater manipulative difficulty when dealing with the more oxidisable and electropositive metals no attempt has been made to push these investigations with potassium-sodium alloy very far. The primary object in view in conducting them was to ascertain if the photoelectric effects could be utilized as an oscillation detector in radiotelegraphy; but as a type of glow-lamp detector has now been found by the writer far more efficient than that originally proposed by him, involving the employment of a carbon filament, these photoelectric experiments have not been pursued. They are put on record here merely for the sake of aiding any who may wish to show them as interesting lecture or class experiments, or pursue the purely physical investigation of the effect itself still further.

The question of photoelectric effects is not without interest in connexion with long distance radiotelegraphy. It has been shown that perfectly dust-free air is not ionized by ultra-violet light. If, therefore, the absorption of long radiotelegraphic electric waves which is found to exist when they pass through considerable distances of sunlit air is due to the presence of free ions in the air, these may arise from the photoelectric action of the light upon the dust particles. This suggests the question whether these particles may not be the same that create the blue colour of the sky. We know that whenever photoelectric effects take place, light must be absorbed, and light of the same kind as that emitted by the

photoelectric substance if it radiates. Thus zinc is rendered photoelectric under the action of ultra-violet light, but zinc when heated, whether by being used as spark-balls for an electric spark or as arc terminals for an electric arc, radiates much ultra-violet light.

In the same manner, the photoelectric effect of potassium appears to be due to the absorption of that violet ray which potassium itself emits if heated. If, then, these atmospheric particles absorb ultra-violet light, that would account for the relatively small percentage of ultra-violet light found in sunlight at the earth's surface, and also for the ionization found to exist in the atmosphere. This suggests the need for further observation on the number of ions present in the terrestrial atmosphere at various heights above the sea-level.

In conclusion, I have pleasure in mentioning the aid rendered in these experiments by my assistant, Mr. G. B. Dyke.

#### DISCUSSION.

Prof. S. P. THOMPSON asked the Author whether Sodium was more sensitive to yellow light than to light of other wave-lengths.

Dr R. S. WILLOWS remarked that the E.M.F. was in the direction required by theory: the alloy emits negative ions, and hence the direction of flow of negative electricity in the cell is from the alloy to the platinum or the alloy is the positive pole. The action was similar in results to those produced by solution pressure when metals were immersed in acids except that the ions produced by light were of the other sign. Prof. Fleming's idea that metals respond to those rays only which they absorb was interesting in view of the fact that the weight of evidence was in favour of the emission of ions being due to an explosion in the atom arising from resonance, but Ladenburg's experiments did not support Dr Fleming's theory. He measured the intensity of the light with a thermocouple, and found for the same intensity that the shorter the wave-length the more efficient was the light in producing emission of ions.

Dr FLEMING, in reply to Prof. Thompson, said Sodium was most sensitive to yellow light.

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XXXI. *A Graphic Method of dealing with Combinations of Co-axial Refracting Surfaces.* By H. STANLEY ALLEN, M.A., B.Sc. Senior Lecturer in Physics at King's College, London \*.

[ABSTRACT.]

IN this paper a graphic method is given that may be applied to the case of two thin lenses not in contact, or to the case of two co-axial refracting surfaces (as in the case of the thick lens). Further the method may be extended to the combination of any two co-axial systems. It affords a way of finding, by a simple geometric construction, the focal length of the combination, and also the positions of the focal planes and principal planes (planes of unit magnification) of Gauss.

Although in actual experiments we are restricted by the formulæ of the first order to rays making small angles with the axis, it is legitimate for purposes of construction to consider rays making finite angles with the axis. By the choice of two particular rays passing through a definite point in the plane of the geometric centre of two refracting systems, the cardinal points of the combination may easily be determined.

In this abstract the method is applied to the general case of the combination of two refracting systems in any media †. We assume that the positions of the focal points and principal planes of each system are given. We can then deduce the existence and positions of the nodal points, defined by the property that an in-ray through one nodal point is parallel to an out-ray through the other nodal point.

Through  $F_1'$ , the second focal point of the first system (fig. 1), draw a line  $F_1'E_1$  at right angles to the axis, equal to the second focal length of that system. Draw  $F_2E_2$  similarly through the first focal point of the second system, equal to the first focal length of that system. Let  $F_1'E_2$  and  $F_2E_1$  intersect in C. Draw CB at right angles to the

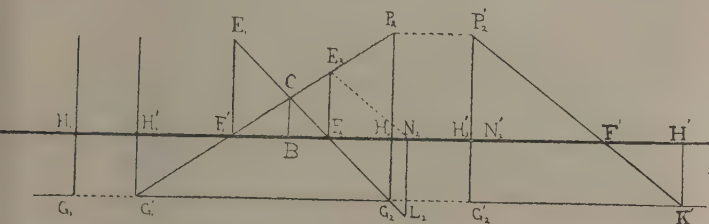
\* Read November 27, 1908.

† Drude, *The Theory of Optics*, p. 28.



axis. The point  $B$  may be called the "geometric centre" or the "optic centre" of the system. It divides the distance between the components in the ratio of their focal lengths. Hence the image of  $CB$  formed by the first system is of the

Fig. 1.



same size as the image of  $CB$  formed by the second system. It follows that these images are planes of unit magnification with regard to the combination, in other words they are the principal planes\*.

The construction proceeds as follows. Let  $CE_1$ , produced if necessary, meet the first principal plane,  $H_2$ , of the second system in  $G_2$ , and the first nodal plane,  $N_2$ , of that system in  $L_2$ . Let  $CE_2$ , produced if necessary, meet the second principal plane,  $H_1$ , of the first system in  $G_1'$ , and the first principal plane of the second system in  $P_2$ . Then  $G_1'G_2$  is parallel to the axis. Through  $P_2$ , a point in the second principal plane of the second system at the same distance from the axis as  $P_2$ , draw a line parallel to  $E_2N_2$  meeting the axis in  $F'$  and  $G_1'G_2$  in  $K'$ . Then  $F'$  is the second focal point of the combination, and  $K'$ , the image of  $C$  formed by the second system, lies in the second principal plane  $H'$  of the combination. The second focal length of the combination is given by  $H'F'$  or by  $N_2L_2$ .

\* "If we imagine a circular aperture or pupil to be placed at the geometric centre then the image of it formed in the first principal plane by  $L_1$  (the first lens) will be exactly equal to the other image of it formed in the second principal plane by  $L_2$  (the second lens). The two principal planes are in this way shown to be planes of unit magnification relatively to one another."—H. Dennis Taylor, "A System of Applied Optics."

I am indebted to my former colleague, Mr. J. Nicol, for drawing my attention to the importance of the optic centre in the general case.

To verify these results trace the course of the rays  $CE_2$ ,  $CF_2$  through the system from left to right.

The in-ray corresponding to  $F_1'C$  passes through  $G_1$  and is parallel to the axis. The out-ray passes through  $P_2'$  and is parallel to  $E_2N_2$ , for it must pass through the image of  $E_2$ , and this image is formed at infinity since  $E_2$  lies in the first focal plane of the second system. This out-ray cuts the axis in  $F'$ , the second focal point of the combination.

The ray  $CF_2$  emerges through  $G_2'$  in a direction parallel to the axis. The intersection of the two out-rays gives the position of  $K'$  the image of  $C$ , in the second principal plane of the combination. Thus  $H'$ ,  $F'$  are determined and  $H'F'$  is the focal length. But by similar triangles

$$\frac{F'H'}{H'K'} = \frac{F_2N_2}{F_2E_2} = \frac{F_2N_2}{F_2H_2}.$$

Therefore

$$F'H' = \frac{F_2N_2}{F_2H_2} \times H_2G_2 = N_2L_2.$$

The first focal point, the first principal plane, and the first focal length can be determined in a similar way.

In the more common case in which both systems are in the same medium, as for two lens systems in air, the construction is still simpler, since the nodal points and principal points coincide. In this case  $F_2N_2 = F_2H_2 = F_2E_2$  so that  $E_2N_2$  makes an angle of  $45^\circ$  with the axis\*.

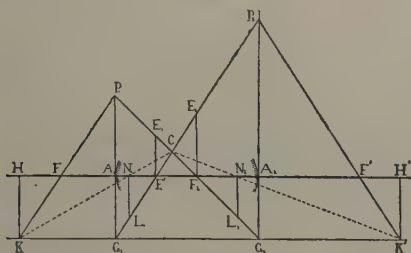
In fig. 2 we have the application of the method to the case of two co-axial refracting surfaces a finite distance apart, as in the thick lens. The principal planes of the first refracting surface coincide at  $A_1$ , the nodal points coincide at  $N_1$  the centre of curvature.

The case of two thin lenses separated by a finite interval is shown in fig. 3. It is to be noticed that  $P_1K$ ,  $P_2K'$  make

\* We may note that if a stop is put at  $B$ , the geometric centre of two lens systems in air, the images of this point formed by the two systems are orthoscopic points, for in this case the images coincide with the nodal points of the combination. (Drude, 'The Theory of Optics,' pp. 64-65).

angles of  $45^\circ$  with the axis, so that the focal length,  $\mathbf{HF}$ , of the combination, is equal to  $\mathbf{HK}$  or to  $\mathbf{A_1G_1}$ .

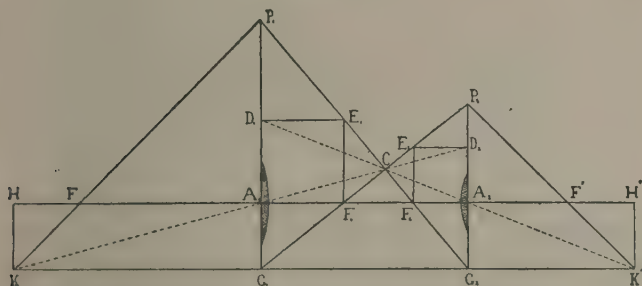
Fig. 2.



Interesting illustrations of the method are found in eye-pieces formed of two lenses, such as those of Ramsden or Huyghens.

It appears to the writer that the method here given may be of value for investigating the way in which the properties of a combination are affected by making given alterations in

Fig. 3.



the character or separation of the components. It may also be of use in enabling non-mathematical students to realise the existence of the cardinal points for a complicated system of refracting surfaces.

### DISCUSSION.

Dr ECCLES remarked that the Author's treatment of refracting surfaces was limited to spherical surfaces and to the direct incidence of

pencils of infinitesimal aperture. If the treatment were extended to the discussion of oblique pencils and the construction of caustics it would be very valuable. The Author's method, as developed to the present, consisted of applying the well-known graphical construction for a harmonic mean, to the successive lenses or spherical surfaces met with in the coaxial system of lenses or surfaces under consideration. Now the harmonic construction just mentioned is a particular case of the general graphical connection between two collinear projective (or homographic) ranges of points; and therefore the speaker preferred, in his own teaching of geometrical optics, to use general projective constructions rather than this harmonic construction. As a fact, the general projective method would be felt to be the easier of the two, by anyone familiar with the fundamental principles of projective geometry, or, as it is sometimes called, "geometry of position." From the point of view of projective geometry the object-points and the image-points on the axis of a refracting (or reflecting surface), since they have a true one-one correspondence (as shown by their algebraic homographic connection), constitute two collinear projective ranges of points. Two such ranges can be graphically correlated, it is known, by at most three projections. The rays used in the first and last of these projections may be interpreted optically as rays of light through a source of light and through its image. The intermediate projection gives the central point on which the Author lays stress. If this point be taken at infinity the planes of unit magnification become defined. Since any number of successive projections at successive surfaces leave the ultimate range of image-points homographic with the original range of object-points, it is proved that any lens system whatever of the kind specified by the Author can be summarised graphically by three projections. Moreover, no graphical construction can be obtained which is not comprehended in the general projective method just sketched. In treatises on projective geometry the properties of these projective ranges and pencils are demonstrated, and they need merely be translated into optical language to give many beautiful and powerful graphical optical theorems. There is reason to think that caustics can be approached along these lines. The speaker expressed the hope that the Author would devote still further effort to these much desired simplifications of geometrical optics.

Dr A. RUSSELL congratulated the Author, and indicated a simple graphical method of determining the focal length of two thin lenses in contact when the focal length of each was known.

Prof. C. H. LEES thought the graphical method described by the Author was an improvement upon algebraic methods.

XXXII. *The Diffusion of Actinium and Thorium Emanations.*

By SIDNEY RUSS, B.Sc., *Demonstrator in Physics, Manchester University* \*.

*Introduction.*

No direct determinations of the molecular weights of the radioactive emanations have yet been made. Owing to the very small quantities of these substances available, experimental work on this subject has been almost completely restricted to a study of the way in which the emanations diffuse, and by a comparison of the diffusion coefficients so obtained with those of gases of known molecular weights, values of the molecular weights for the emanations themselves may be inferred by means of Graham's law.

The work of Curie and Danne †, Rutherford ‡, Rutherford and Miss Brooks §, and Makower || on these lines has indicated, for the emanations of radium and thorium, a molecular weight of the order one hundred. Bumstead and Wheeler ¶, however, give a rather higher value in the case of radium, while for the emanation of actinium, Debierne \*\* obtained a molecular weight of seventy.

On the disintegration theory radium and thorium emanations should have very approximately the same molecular weight, namely, 222. There is thus a very large discrepancy between the theoretical and experimental numbers.

In order to account for this it has been suggested that the ordinary laws of diffusion do not perhaps hold when very minute quantities of the substances in question are being dealt with. There being no obvious remedy in this direction, it was thought that by a study of the diffusion of the emanations in gases differing considerably in molecular weight, any existing discrepancies would be brought to light. With

\* Read November 27, 1908.

† *Comptes Rendus*, 1903, p. 114.

‡ 'Radioactivity,' p. 276.

§ *Trans. Roy. Soc. of Canada*, 1901-2.

|| *Phil. Mag.* 1905, p. 56.

¶ *Le Radium*, June 1907.

\*\* *American Journal of Science*, Feb. 1904.



this object in view, the diffusion coefficients of actinium emanation in air, hydrogen, carbon dioxide, and sulphur dioxide were determined. While the work was in progress there appeared in the American Journal of Science, June 1908, a paper by P. B. Perkins, in which the diffusion of radium emanation was compared directly with that of mercury vapour, the experiments giving a molecular weight of 235 for the emanation, a number approximating closely to that predicted by theory. The results of these experiments suggested that it might be advisable to determine the coefficient of diffusion of actinium emanation, itself presumably monatomic, into a monatomic gas. For this purpose a quantity of argon was prepared, and the coefficient obtained by two methods

Whatever the nature of the supposed deviations from the ordinary laws of diffusion referred to above, it might reasonably be expected that they would be exhibited to an equal extent by the different emanations. A direct comparison was therefore carried out over a fairly wide range of pressure between the diffusion of actinium and thorium emanations, a comparison of their coefficients leading directly to the ratio of their molecular weights.

#### *Method of Experiment.*

The theory of the method here adopted for obtaining the diffusion coefficient has been given by Rutherford\*. If a layer of radioactive material be placed at the bottom of a cylindrical vessel a gradient of the emanation is set up, the partial pressure at any layer distant  $x$  from the base being given by  $p = p_0 e^{-\sqrt{\frac{\lambda}{K}} \cdot x}$ , where  $\lambda$  is the radioactive constant of the emanation and  $K$  its diffusion coefficient. It is not convenient in practice to measure the partial pressure of the emanation, but if a metal rod be placed along the axis of the cylinder in question and kept at a high negative potential, the active deposit produced by the emanation is directed to the rod soon after formation. We have thus a layer of active matter on the rod, the gradient of which is that of

\* 'Radioactivity,' p. 275.

the emanation within the cylinder. The distribution of active matter along the rod can be found by measuring the ionization produced by the  $\alpha$  rays emitted by successive small segments of the rod. This gives us the variation of  $p$  with  $x$  in the above equation, and knowing the value of  $\lambda$ , that of  $K$  is determined.

Debiegne\* has shown that the above theory is also applicable when the emanation is allowed to diffuse up into the space between two parallel plates. In an exactly analogous way a deposit of active matter is obtained on the plates, the gradient of which is that of the emanation itself.

Both of these experimental arrangements have been used in the course of this work.

#### *Actinium Emanation in Different Gases.*

For determining the coefficient of diffusion of the emanation in different gases the following experimental arrangement was used. A cylindrical vessel 12.5 cms. long and 7.5 cms. diameter was fitted with a base on which was placed a tray containing a preparation of actinium; about 1 cm. above the surface of the tray which was covered with tissue paper, were suspended vertically two glass plates 10 cms. long, 2.2 cms. broad, and .2 cm. thick, which were kept separated .2 cm. from one another. The plates were silvered and metallic connexions supplied, so that an electric field might be applied between the two plates if necessary. It may be stated here that the gradients obtained on the plates with, and without, an electric field were identical, the only difference being that nearly all of the activity was concentrated on the negative plate when the field was applied, while it was shared equally between the two plates with no field.

The gases before entering the diffusion vessel were passed through calcium chloride and cotton-wool tubes to render them dry and dust free. The vessel was placed inside a large tank of water, which prevented any appreciable change of temperature occurring during an experiment. During the whole course of the work the temperature varied between 10° and 18° C.

\* *Le Radium*, June 1907

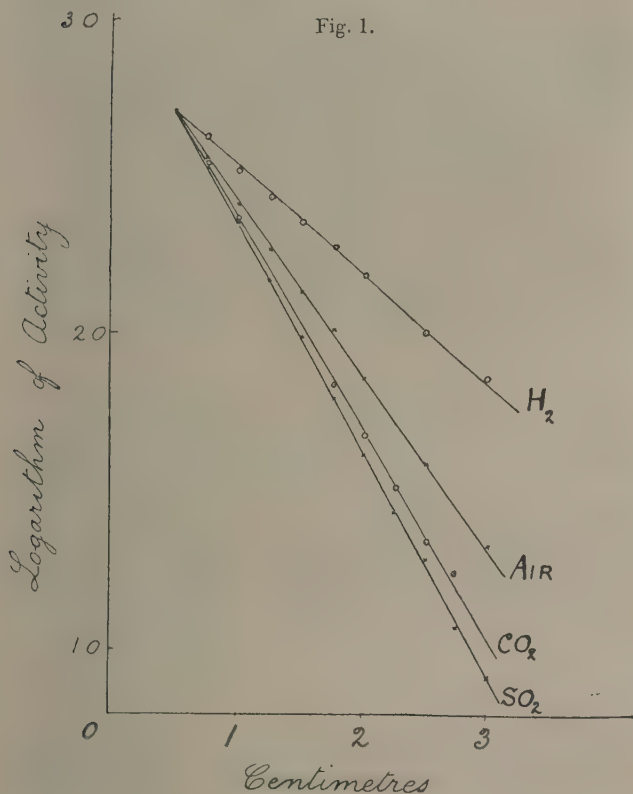
The glass plates were usually exposed for about twenty-four hours, and then they were removed in order to measure the gradient of active deposit on their surfaces. One of the plates was placed on a small platform which could be moved along so that small portions ( $\cdot 4$  cm. lengths) of it came beneath a slit in a lead sheet, just above which was an ionization vessel. The saturation current produced in this vessel by the  $\alpha$  rays emitted from the active deposit on the plate was measured by means of an electrometer, and readings were usually taken for about 4 cms. length of the plate. Corrections were applied for the decay of the active deposit with time, for change in sensitiveness of the electrometer during the measurements, and for the natural leak of the ionization vessel.

A selection of the results for the different gases will be found in Table I. and seen graphically in fig. 1, where the abscissæ represent distances along the plate and the ordinates the logarithm of the corresponding activity. For convenience in representing the results the initial ordinate has always been taken through the same point. The mean of three

TABLE I.—Diffusion of Actinium Emanation.

Distance in cms.	<i>Air.</i> Log. of Activity.	<i>Hydrogen.</i> Log. of Activity.	<i>Carbon Dioxide.</i> Log. of Activity.	<i>Sulphur Dioxide.</i> Log. of Activity.
$\cdot 5$ .....	2.341	2.194	2.581	2.709
$\cdot 75$ .....	2.182	2.115	2.416	2.526
1.0 .....	2.036	1.997	2.233	2.358
1.25 .....	1.901	1.919	2.026	2.173
1.5 .....	1.768	1.837	1.846	1.992
1.75 .....	1.648	1.761	1.710	1.798
2.0 .....	1.494	1.668	1.531	1.616
2.25 .....	...	...	1.391	1.439
2.5 .....	1.224	1.496	1.212	1.281
2.75 .....	...	...	1.112	1.068
3.0 .....	.964	1.351	...	.912
<i>Diffusion coefficient.</i>	Mean of three	Mean of two	Mean of two	Mean of two
K Experimental...	.096	.330	.073	.062
K Calculated .....	...	.363	.078	.064

values of the coefficient for air was  $\cdot 096$ , which is somewhat lower than  $\cdot 112$ , the value found by Debierne. In the light of subsequent work with different diffusion vessels, which



gave higher values, it is thought that the rather low value obtained is due to the fact that the preparation of actinium did not completely cover the base of the diffusion vessel, this would cause some of the emanation to diffuse to the sides of the vessel, and consequently make the vertical gradient too

steep. This, however, does not interfere with a comparison being made in the different gases.

According to Graham's law the diffusion coefficient of a gas is inversely proportional to the square root of the molecular weight of the gas into which it is diffusing. Hence, taking the value obtained for the diffusion coefficient of the emanation into air, we may calculate its coefficient when diffusing into the other gases which have been worked with. This has been done and the calculated values given at the bottom of Table I. (p. 488).

It will be seen that the differences between the experimental and calculated values get larger as we go from a heavy gas to a light one; but that even for hydrogen there is agreement to within about 10 per cent., so that one may say that if discrepancies do exist in the diffusion processes, they are exhibited fairly equally by these different gases.

#### *Actinium Emanation in Argon.*

As already stated, it was thought advisable to observe the behaviour of the emanation when diffusing into a monatomic gas; for this purpose argon was chosen. With the kind assistance of Mr. J. N. Pring about 400 cubic centimetres of the gas were prepared, by passing air, first over red hot copper to remove the oxygen, and then over calcium, heated strongly in an iron tube, to absorb the nitrogen. After passing the gas repeatedly over the calcium a density determination gave the value 19.7, which indicated a purity sufficient for the purposes of the experiment. (Density of argon = 20.)

Owing to the small quantity of gas available smaller cylindrical diffusion vessels were made; one with two parallel glass plates suspended above some actinium, spread so as to completely cover the base of the vessel, and the other fitted with an ebonite plug 3 cms. long resting on the base of the vessel. A uniform layer of actinium was spread over the plug, a hole through the centre of which served to keep a metal rod along the axis of the cylinder, which was made of brass. The metal rod consisted of small seg-



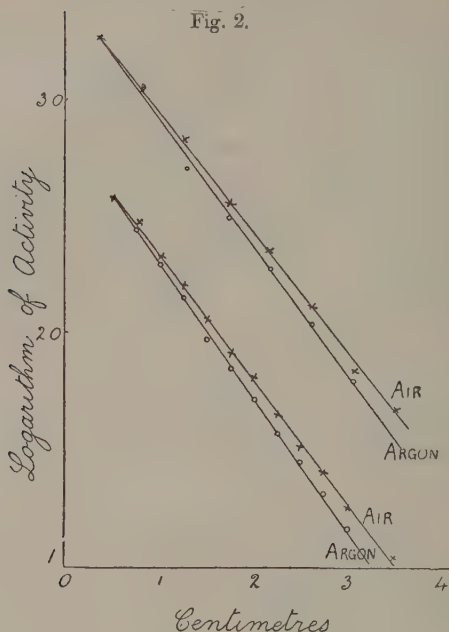
ments of brass sliding on a central steel rod. On applying an electric field between the rod and containing vessel, the active deposit was directed to the rod and the gradient determined by placing each of the small brass segments successively into an  $\alpha$  ray electroscope and measuring the ionization thereby produced. The gradient on the plates was determined in the manner already described. The results of the two methods of measurement are given in Table II. and seen in fig. 2 (p. 492), where the gradients obtained with the segmented rod are drawn above those measured from the glass plates.

TABLE II.

Diffusion of Actinium Emanation in :—

Distance in cms.	<i>Air.   Argon.</i> <i>Parallel Plates.</i>		Distance in cms.	<i>Air.   Argon.</i> <i>Segmented Rod.</i>	
	Log. of Activity.	Log. of Activity.		Log. of Activity.	Log. of Activity.
.5 .....	2.550	2.574			
.75 .....	2.454	2.438			
1.0 .....	2.301	2.297	1.37 .....	2.609	2.472
1.25 .....	2.178	2.143	1.82 .....	2.380	2.260
1.5 .....	2.033	1.983	2.27 .....	2.153	1.897
1.75 .....	1.885	1.841	2.73 .....	1.885	1.686
2.0 .....	1.781	1.713	3.18 .....	1.682	1.470
2.25 .....	1.623	1.582	3.62 .....	1.438	1.239
2.5 .....	1.489	1.455	4.07 .....	1.163	.990
2.75 .....	1.382	1.305	4.53 .....	1.014	
3.0 .....	1.231	1.175			
3.5 .....	1.020	.956			
4.0 .....	.896	.612			
Diff. coeff. K	Mean of three			Mean of two	
Experimental .....	.118	.106		.122	.109
Calculated ... ..	...	.100		...	.104

The coefficients obtained by the two different methods agree well with one another, also with the numbers calculated from Graham's law, making use of the numerical values obtained for air.



*Variation of the Coefficient with Pressure.*

According to the kinetic theory of gases the coefficient of diffusion is inversely proportional to the total pressure of the two diffusing gases; the product of the coefficient and the pressure should therefore remain constant.

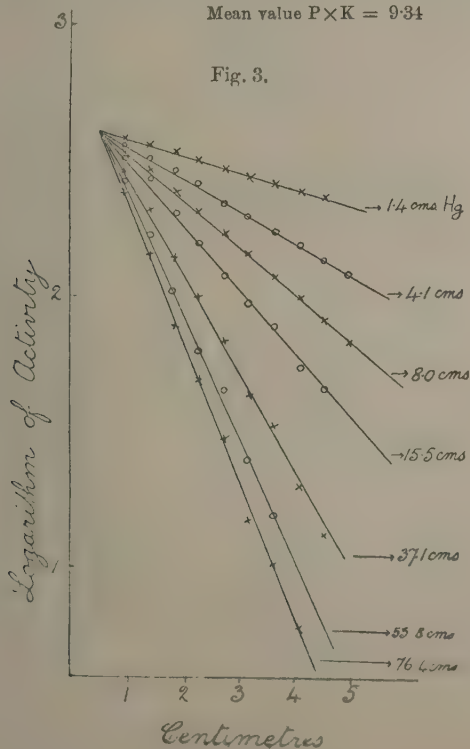
The variation of the coefficient has been determined for pressures between 76.4 cms. and 1.4 cm.; the small vessel containing the segmented rod was used. The results are recorded in Table III. & fig. 3. It will be seen that over this range of pressure, with the exception of 1.4 cm. pressure, the product  $P \times K$  is approximately constant, its mean value being 9.34. It is difficult to say whether the high value 10.9 at the lowest pressure indicates a deviation from the above law as the gradient obtained is a very small one, and relatively large errors may be made in estimating it.

TABLE III.—Actinium Emanation in Air.  
Variation of Diffusion Coefficient with Pressure.

	1.4 cms.Hg.	4.1 cms.	8.0 cms.	15.5 cms.	37.1 cms.	55.8 cms.	76.4 cms.
Distance in cms.	Log. of Activity.	Log. of Activity.	Log. of Activity.	Log. of Activity.	Log. of Activity.	Log. of Activity.	Log. of Activity.
.46 .....	1.618	1.797	1.850	1.254	1.518	2.246	2.609
.92 .....	1.601	1.744	1.759	1.157	1.367	2.057	2.380
1.37 .....	1.564	1.689	1.704	1.074	1.230	1.860	2.153
1.83 .....	1.548	1.646	1.625	.946	1.044	1.650	1.885
2.28 .....	1.509	1.596	1.548	.829	.909	1.434	1.682
2.72 .....	1.485	1.523	1.469	.712	.745	1.286	1.438
3.17 .....	1.445	1.477	1.391	.611	.539	1.024	1.163
3.63 .....	1.425	1.421	1.313	.528	.428	.829	1.014
4.09 .....	1.389	1.380	1.232	.374	.204	.734	.784
4.55 .....	1.375	1.316	1.147	.298	.020		
5.01 .....	...	1.268	1.071				
Diff. coeff. K	7.81	2.37	1.16	.582	.248	.167	.125
Pressure $\times$ K	10.9	9.7	9.3	9.0	9.2	9.3	9.5

Mean value  $P \times K = 9.34$

Fig. 3.



*Diffusion of Thorium Emanation in Air and Argon.*

Since the time-periods of thorium and actinium emanations are of the same order of magnitude, and previous work (*loc. cit.*) has shown their diffusion coefficients to be similar, the opportunity presented itself of comparing these two emanations under precisely the same conditions of diffusion.

The small diffusion vessels which had been used for actinium were emptied of their preparations and thoria substituted. The measurements were conducted in a similar way, except that the  $\alpha$  ray electroscope could not now be used, as the thoria being considerably weaker in emanating power than the actinium, the amount of active deposit collected was smaller. The gradient along the metal rod was consequently measured by means of a rather sensitive electrometer in the manner already described for the glass plates.

Observations were made in air between 76 cms. and 8.25 cms. pressure, and in argon at 76 cms.

The results are placed together in Table IV. and may be

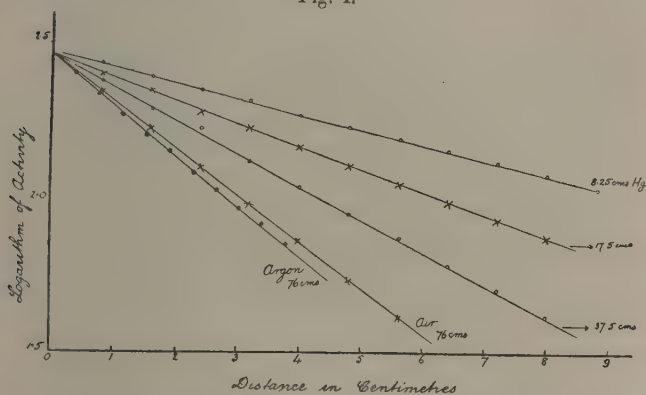
TABLE IV.  
Diffusion of Thorium Emanation.

Pressure ...	AIR.				ARGON.	
	8.25 cms.	17.5 cms.	37.52 cms.	76.12 cms.	76 cms.	
Distance in cms.	Log. of Activity.	Log. of Activity.	Log. of Activity.	Log. of Activity.	Distance in cms.	Log. of Activity.
0 .....	2.401	2.472	2.373	2.568	0 .....	1.875
.8 .....	2.375	2.406	2.282	2.453	.38 .....	1.813
1.6 .....	2.327	2.354	2.192	2.329	.76 .....	1.741
2.4 .....	2.289	2.285	2.134	2.203	1.14 .....	1.672
3.2 .....	2.255	2.239	2.028	2.082	1.52 .....	1.608
4.0 .....	2.206	2.176	1.947	1.962	1.90 .....	1.563
4.8 .....	2.169	2.116	1.853	1.833	2.28 .....	1.490
5.6 .....	2.131	2.058	1.775	1.713	2.66 .....	1.434
6.4 .....	2.094	1.989	1.685		3.04 .....	1.369
7.2 .....	2.053	1.937	1.602		3.42 .....	1.327
8.0 .....	2.014	1.874	1.517		3.80 .....	1.254
8.8 .....	1.969					
Diff. coeff. K	.966	.436	.211	.103	K experimental	.084
Pressure $\times$ diff. coeff. (P $\times$ K) ..	7.97	7.63	7.94	7.88	K calculated ...	.087

Mean value  $P \times K = 7.85$

seen in fig. 4, where the abscissæ represent the distances along the rod and the ordinates the logarithm of the corresponding activity.

Fig. 4.



It will be seen from the last row of Table IV. that the product, pressure  $\times$  diffusion coefficient ( $P \times K$ ) remains sensibly constant, the mean value being 7.85.

As in the case of actinium emanation no large deviation from ordinary gas laws is observable when thorium emanation diffuses into a monatomic gas of comparatively low molecular weight like argon; the experimental value obtained for the diffusion coefficient in this case being very nearly that calculated by means of Graham's law.

#### *Comparison of Actinium and Thorium Emanation.*

A direct comparison may now be made between the diffusion coefficients of these two emanations, which leads to a ratio of their molecular weights.

Taking the mean value of  $P \times K$  for actinium emanation in air as 9.34, and for thorium as 7.85, and dividing each by 76 to reduce to atmospheric pressure, we have:—

$$\frac{\text{Diffusion coefficient of actinium emanation in air}}{\text{Diffusion coefficient of thorium emanation in air}} = \frac{.123}{.103} = 1.19$$

which gives the ratio:—

$$\frac{\text{Molecular weight of thorium emanation}}{\text{Molecular weight of actinium emanation}} = (1.19)^2 = 1.42.$$



It is interesting to compare this ratio with that which is obtained by taking the values of the diffusion coefficients of thorium and actinium emanations which have been determined by other observers under different experimental conditions.

For thorium emanation we have two sets of experiments; those of Rutherford\* giving a diffusion coefficient  $\cdot 09$  and those of Makower† giving  $\cdot 109$ . In the case of actinium emanation there has, up till now (as far as the author is aware), been only one determination of its diffusion coefficient, namely, that of Debierne‡, who obtained the value  $\cdot 112$ . Taking the mean of the two values quoted for thorium as  $\cdot 99$  we obtain for the ratio of the molecular weights  $\left(\frac{\cdot 112}{\cdot 99}\right)^2 = 1\cdot 28$ .

It will be seen, on reference to Tables III. and IV., that the *mean value* of  $P \times K$  does not show as much as 5 per cent. variation from any one of the values themselves, this would indicate that the number given for the ratio of the molecular weights,  $1\cdot 42$  is not subject to as much as a 10 per cent. error.

The conclusion arrived at that actinium emanation is of considerably lower molecular weight than thorium (and therefore also than radium) emanation, is quite in agreement with much of the recent work which has been done, showing that actinium is not one of the products in the *direct* line of descent from uranium to radium §.

### Summary.

1. The diffusion of actinium emanation in gases such as air, hydrogen, carbon dioxide, sulphur dioxide, and argon shows no considerable deviations from the ordinary laws of diffusion.

2. The variation with pressure of the coefficients of diffusion of the actinium and thorium emanations appears to be quite regular down to pressures of a few centimetres.

\* 'Radioactivity,' p. 276.

† Makower, Phil. Mag. 1905, p. 56.

‡ Debierne, *Le Radium*, 1907, p. 213.

§ Rutherford, Phil. Mag. 1907, vol. xiv. p. 733; Boltwood, 'Nature,' 1907, p. 544.

3. A comparison of the diffusion coefficients of the actinium and thorium emanations in air under similar experimental conditions gives the ratio of their molecular weights as 1.42, that of thorium being the heavier.

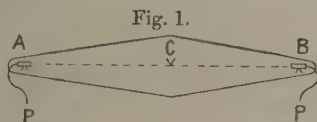
In conclusion I have much pleasure in thanking Professor Rutherford for several suggestions during the course of the work, and Mr. J. N. Pring for assistance in preparing the argon.

#### DISCUSSION.

Prof. C. H. LEES pointed out that the low values for the molecular weights of the emanations had all been obtained by allowing the emanations to diffuse into light gases. Prof. Pickering had allowed them to diffuse into mercury vapour, and obtained a value about 230.

#### XXXIII. *An Accurate Method of Measuring Moments of Inertia.* By the late Professor W. R. CASSIE\*.

IF ACB (fig. 1) is the beam of a balance with moment of inertia  $I$  about C,



length of arm  $AC = BC = a$ ,

coefficient of directive couple  $L$ ,

and scale pans, each of mass  $P$ , hung from the knife edges,

then the time of a small oscillation is

$$t_1 = 2\pi\sqrt{\frac{I + 2Pa^2}{L}}.$$

Here  $Pa^2$  is a quantity of the nature of a moment of inertia involving only a mass and a length, quantities which can be accurately measured. Since any body suspended from the knife-edge does not rotate, the distribution of mass in it does not matter.

The method consists in taking as standard moment of

\* Read November 27, 1908.

inertia a known mass hung from the knife-edge of a balance, and comparing others with it.

For instance, if  $t_2$  were the period with a mass  $Q$  in each scale-pan,

$$Qa^2 = (L/8\pi^2)(t_2^2 - t_1^2).$$

Now if without altering  $L$  we could attach to the beam a bar of moment of inertia  $I'$  (unknown), and the period with  $I'$  attached to the beam, and the scale-pans only on the knife-edges, were  $t_3$ , then

$$I' = (L/8\pi^2)(t_3^2 - t_1^2),$$

so that

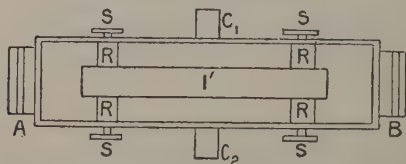
$$I' = Qa^2(t_3^2 - t_1^2)/(t_2^2 - t_1^2).$$

The value of  $I'$  is thus obtained in terms of quantities which can be accurately measured.

There remains the practical problem of attaching the bar to the beam without altering  $L$ . This can be done as follows.

Make the beam in the form of a trough (for the plan see fig. 2) so that the bar  $I'$  may be laid inside it.  $A$  and  $B$

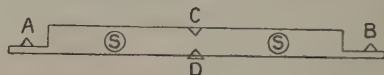
Fig. 2.—Plan of beam.



are the knife-edges for the scale-pans and  $C_1C_2$  is the fulcrum, consisting of two portions projecting from the two sides of the trough.

Now we have to make the centre of gravity of  $I'$  coincide with the knife-edge of the fulcrum  $C_1C_2$ . To this end

Fig. 3.—Elevation of beam.



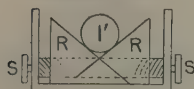
ACB being in one plane (see fig. 3), make the centre of

gravity of the beam come on the knife-edge C (by adjusting a gravity bob), and apply the directive couple by hanging a known weight  $Wg$  from the knife-edge D parallel to but below C. Then with the weight  $Wg$  removed and the beam in neutral equilibrium, lay  $I'$  in the trough and adjust its position until the beam is again in neutral equilibrium. Then the centre of gravity of  $I'$  is also in the knife-edge C, and when  $Wg$  is replaced the directive couple  $L$  is unchanged.

This adjustment of the position of  $I'$  must be made without moving the centre of gravity of the beam or changing  $I$ , the moment of inertia of the beam about C. This may be done as follows (see fig. 4):—

Let the bar  $I'$  lie on two rests each consisting of two wedge-shaped pieces  $RR$  forming a V, and let these wedges be moved simultaneously and equally in opposite directions by the right- and left-hand screw-threads cut in the bar whose ends  $SS$  appear outside the trough of the beam. By this movement of  $RR$  the

Fig. 4.  
Section of beam.



height of  $I'$  may be altered until the equilibrium is neutral. Now these movements are entirely parallel to the fulcrum  $C_1C_2$ , and therefore do not alter either  $I$  or the position of the C.G. of the beam.

Thus the process for measuring the moment of inertia of a bar would be as follows:—

1. Bring the balance with weight  $Wg$  removed to neutral equilibrium.
2. Hang on  $Wg$  and take the period,  $t_1$ , of a small oscillation.
3. Put equal weights  $Q$  in the scale-pans and take the period,  $t_2$ .
4. Remove the weights  $Q$  and  $Wg$  and put the bar  $I'$  in the trough, and adjust again to neutral equilibrium by means of the screwheads  $SS$ .
5. Hang on  $Wg$  and take the period,  $t_3$ .
6. Measure  $a$ , the length of the arm,

then 
$$I' = 2Qa^2(t_3^2 - t_1^2)/(t_2^2 - t_1^2).$$

## DISCUSSION.

The SECRETARY read a letter from Prof. W. STROUD describing a method of determining moments of inertia in use in the University of Leeds. In this method two vibrators perpendicular to one another are used, and they can be arranged to produce a Lissajou figure. The horizontal vibrator carries a vertical slit and *vice versa*; and as the two slits are placed one behind the other, a small spot of light can be seen through the combination. The vertical vibrator consists of a stout metal framework supported by a torsion wire, and carrying a platform upon which the body whose moment of inertia is required can be placed. The time of swing of this vibrator is then adjusted (by the addition of rings of known moment of inertia) to be equal to that of the horizontal vibrator, so that the Lissajou figure is invariable. The body is then placed upon the platform, and inertia rings are removed till the Lissajou figure is reproduced or nearly so. Then, accurate observations are taken of the number of oscillations required to complete a cycle with inertia rings slightly below and slightly above the correct value, which is got by interpolation. The vibrators are kept in continuous oscillation by an electromagnetic make and break.

Prof. C. H. LEES said that Prof. Stroud's method did not seem capable of giving very accurate results on account of want of homogeneity of the rings.

Dr. A. RUSSELL asked if it was necessary to take account of changes in barometric pressure in accurate work on moments of inertia.

The CHAIRMAN remarked that in delicate work attention must be paid to the effects of changes in barometric pressure.

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XXXIV. *On the Lateral Deflexion and Vibration of "Clamped-Directed" Bars.* By JOHN MORROW, M.Sc. (Vict.), D. Eng. (L'pool), Lecturer in Engineering, Armstrong College (University of Durham) \*.

Section I.—*Introduction and Contents.*

§ 1. THE vibrations of a bar under the terminal conditions which are probably most frequent in engineering practice, appear to have hitherto attracted but little attention. These conditions occur when one end of the bar is clamped, and the other is constrained to retain its original direction. Such conditions may be realized by having two initially parallel

\* Read June 12, 1908.



bars, CA and DB (fig. 1), each clamped at one end, with the otherwise free ends connected by a rigid bar AB.

Figs. 1 and 2 show two distinct ways in which such a

Fig. 1.

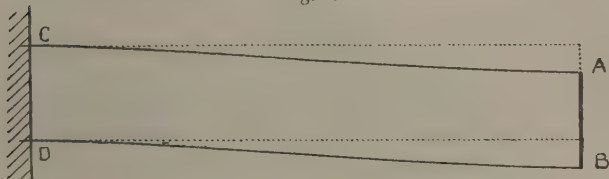
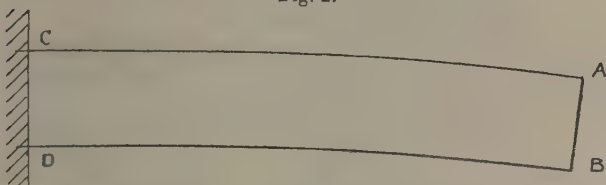


Fig. 2.



system may vibrate. In the former case we have pure lateral vibration, whereas in the latter there is a necessary accompaniment of longitudinal vibrations in each bar. The former case is the more important since in it the frequency of the fundamental type will, in general, be the lower.

I propose to refer to a bar under the end conditions of fig. 1 as a "Clamped-Directed" bar.

Although there exists a considerable literature on the theory of the lateral vibration of thin rods, the practical application of the results is extremely limited. The explanation of this is to be sought, not in any lack of enterprise on the part of the technologist, but rather in the fact that the problems that have been solved are not necessarily those of which the solutions are most required in practice.

§ 2. As an important example, in which the terminal conditions here considered occur, one might mention the case of the cylinder of a steam-engine supported on two or more mild steel standards. It will be recognized that the upper ends of the standards must be treated as "directed."

It is worthy of notice that these terminal conditions are mentioned by Lord Rayleigh in his 'Theory of Sound'\*

\* See Rayleigh's 'Sound,' vol. i. p. 259, 1894 edition.

but that the directed end is at once dismissed from consideration with the remark that "there are no experimental means by which the contemplated constraint could be realized."

§ 3. *Notation and End Conditions.*—

Let  $y, y_z$  = deflexions at points  $x$  and  $z$  in the length of the bar ;

$\omega, I$  = area and moment of inertia of cross-section, the cross-dimensions being supposed small compared with the length ;

$l$  = length of bar ;

$\rho$  = density of the material ;

$E$  = Young's Modulus for the material (when there is an axial pull  $P$  in the bar,  $E$  stands for  $P/\omega + \text{Young's Modulus}$ ) ;

$t$  = time, measured from any instant at which  $y$  is everywhere zero ;

$N$  = number of complete vibrations per second.

The symbol  $\ddot{y}$  is written for  $d^2y/dt^2$  ; and if  $y_1$  is the instantaneous deflexion at some particular point in the length of the bar, we have, for simple harmonic vibrations,

$$-\ddot{y}/y = -\ddot{y}_1/y_1 = k^2 \text{ (say), } \quad . \quad . \quad . \quad . \quad (1)$$

where

$$N = k/2\pi.$$

The curve assumed by the elastic central line at any instant is given in terms of  $y_1$ . If  $a$  is the amplitude at the point in the length at which  $y_1$  is the instantaneous deflexion, the value of  $y_1$  is given by

$$y_1 = a \sin kt. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The origin will always be taken at the clamped end. At  $x=0$  we have, therefore,  $y = dy/dx = 0$  ; whilst at the directed end,  $x=l$ ,  $dy/dx=0$ , and, if there be no concentrated load there,  $d^3y/dx^3=0$ .

Section II.—*Unloaded Massive Bar.*

§ 4. In the ordinary case of a bar vibrating under the effect of its own mass only, the form of the elastic central line and the frequency of the lateral vibrations are to be

determined from the well-known equation

$$\frac{d^4 y}{dx^4} = \mu^4 y,$$

in which

$$\mu^4 = -\frac{\rho \omega^2}{EI} \frac{y_1}{y_1},$$

and of which the general solution is

$$y = A \sin \mu x + B \cos \mu x + C \sinh \mu x + D \cosh \mu x.$$

The end conditions give four equations whose compatibility requires that

$$\tan \mu l = -\tanh \mu l, \quad . \quad . \quad . \quad (3)$$

of which the least root is

$$\mu l = 2.36502 \dots \dots \dots (4)$$

The frequency of the fundamental is therefore

$$N = \frac{5.5933}{2\pi} \sqrt{\frac{EI}{\rho \omega l^4}}.$$

§ 5. It can be shown that the values of  $\theta$  satisfying

$$\cos \theta \cosh \theta = 1, \quad . \quad . \quad . \quad (5)$$

are the same as those which satisfy

$$\tan \frac{\theta}{2} = \pm \tanh \frac{\theta}{2},$$

and hence that the admissible roots of equation (3) are each one half of the alternate roots of (5). Accurate solutions of (5) have been provided by Rayleigh\*, and from these the values of  $\mu l$  which satisfy (3) are

$$\mu_1 l = 2.365020$$

$$\mu_2 l = 5.497804$$

$$\mu_3 l = 8.639380,$$

after which

$$\mu_n l = (n - \frac{1}{4})\pi,$$

to more than six decimal places,  $n$  being an integer.

§ 6. The values of  $\mu l$  given above for the lower harmonics may be readily calculated by the method explained below for  $\mu_2 l$ .

\* 'Theory of Sound,' Rayleigh, vol. i. p. 278, 1894 edition.

We know that  $\mu_2 l = (2 - \frac{1}{4})\pi$  approximately. Hence

$$\tanh \mu_2 l = .999967, \quad . \quad . \quad . \quad . \quad (5a)$$

to the limits of accuracy of the tables at my disposal. By equation (3) therefore

$$\tan \mu_2 l = -.999967 \text{ approximately.}$$

$$\therefore \mu_2 l = 2\pi - .7853815 = 5.4978038,$$

which is more than sufficiently accurate\*. Even when this method is used for the fundamental, the error is less than 0.01 per cent.

§ 7. To examine the curve assumed by the centre line of the bar, we obtain a further relation by putting  $y = y_1$  when  $x = l$ ; whence it can easily be shown that

$$y = \frac{1}{2}y_1[(\sinh \mu l + \sin \mu l)(\sin \mu x - \sinh \mu x) - (\cosh \mu l - \cos \mu l)(\cos \mu x - \cosh \mu x)] \div (1 - \cos \mu l \cosh \mu l). \quad (6)$$

The values of  $\mu l$  to be used in this equation are those given in § 5.

§ 8. When dealing with harmonics the positions of the nodes are to be found from (6) by putting  $y = 0$ . We thus get, at each node,

$$\frac{\sin \mu x - \sinh \mu x}{\cos \mu x - \cosh \mu x} = \frac{\cosh \mu l - \cos \mu l}{\sinh \mu l + \sin \mu l}. \quad . \quad . \quad (7)$$

The fundamental is free from nodes. For the first harmonic the value of the right-hand side of (7) is 1.000033. For higher tones it may be taken as unity.

Solving (7) by trial, we find in the case of the first harmonic that the distance of the node from the clamped

\* This method depends on the fact that a slight difference between two numbers of the magnitudes involved makes no appreciable difference in their hyperbolic tangents. If  $h$  is small

$$\tanh(x+h) - \tanh x = h(1 - \tanh^2 x)$$

approximately. In the example in the text the result shows that

$$\tanh \mu_2 l = \tanh(2 - \frac{1}{4})\pi$$

to at least eight decimal places. There can be no hesitation, therefore, in accepting equation (5a). The method appears more natural than the normal one, which would be to put  $\mu_2 l = (2 - \frac{1}{4})\pi + x$  in  $\tan \mu_2 l$  and  $\tanh \mu_2 l$  (where  $x$  is small), and to proceed by approximations.

end is given by

$$x = .7166 l.$$

For higher harmonics, the  $i$ th tone has  $i-1$  nodes; and of these the first node, that is the one nearest the clamped end, is at

$$\frac{x}{l} = \frac{5.0175}{4i-1};$$

the second occurs at

$$\frac{x}{l} = \frac{8.9993}{4i-1},$$

and beyond this the  $j$ th node is given with sufficient accuracy by

$$\frac{x}{l} = \frac{4j+1}{4i-1}.$$

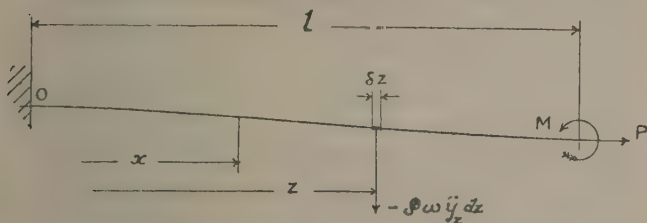
### Section III.—*Unloaded Massive Bar, Longitudinal Tension.*

§ 9. When the bar is subjected to an axial tensile force  $P$  as shown in figure 3, the differential equation, based on the ordinary Bernoulli-Eulerian theory, may be written

$$EI \frac{d^2 y}{dx^2} + \rho \omega \int_x^l \ddot{y}_z (z-x) dz + P(y_1 - y) + M = 0. \quad (8)$$

in which  $M$  is the couple required to direct the end and  $y_1$  is the deflexion there.

Fig. 3.



Differentiating twice with respect to  $x$ ,

$$EI \frac{d^4 y}{dx^4} + \rho \omega \ddot{y} - P \frac{d^2 y}{dx^2} = 0.$$

which is the differential equation common to all unloaded bars of constant flexural rigidity. It may be written

$$\frac{d^4 y}{dx^4} + \frac{\rho \omega}{EI} \ddot{y} - \frac{P}{EI} \frac{d^2 y}{dx^2} = 0, \quad (9)$$



for the case of stationary simple harmonic vibrations. The general solution is then of the form

$$y = A \cosh \alpha x + B \sinh \alpha x + C \cos \beta x + D \sin \beta x,$$

in which, whatever the end conditions,

$$\left. \begin{aligned} -\frac{\ddot{y}_1}{y_1} &= \frac{\alpha^2}{\rho\omega} (EI\alpha^2 - P) \\ -\frac{\ddot{y}_1}{y_1} &= \frac{\beta^2}{\rho\omega} (EI\beta^2 + P) \end{aligned} \right\} \dots \dots \dots (10)^*$$

The terminal conditions give four equations for which it is necessary that

$$\alpha \tanh \alpha l = -\beta \tan \beta l. \quad \dots \dots (11)$$

Elimination of  $\alpha$  and  $\beta$  from the equations (10) and (11) would give an expression for  $-\frac{\ddot{y}_1}{y_1}$  and enable the frequency to be calculated.

§ 10. The form of the centre line of the bar at any instant is then given by

$$y = y_1 (\tanh \alpha l \sinh \alpha x - \cosh \alpha x - \frac{\alpha}{\beta} \tanh \alpha l \sin \beta x + \cos \beta x) \\ \div \left( \frac{1}{\cos \beta l} - \frac{1}{\cosh \alpha l} \right),$$

whilst the couple  $M$  at the directed end can be obtained from

$$\frac{M}{y_1} + P = -\rho\omega \frac{\ddot{y}_1}{y_1} \left( \frac{1}{\beta^2} \cosh \alpha l + \frac{1}{\alpha^2} \cos \beta l \right) \div (\cosh \alpha l - \cos \beta l).$$

§ 11. Failing an exact solution of equations (10) and (11), a useful approximate one can be obtained by limiting the longitudinal force to such values as may occur in practice.

Writing  $\phi = \alpha l$ ,  $\theta = \beta l$ , equation (11) is

$$\phi \tanh \phi = -\theta \tan \theta.$$

$$\therefore \frac{d\phi}{d\theta} = -\frac{\theta + \tan \theta + \theta \tan^2 \theta}{\phi + \tanh \phi - \phi \tanh^2 \phi}.$$

Also (10) gives

$$\theta^2 = \phi^2 - \frac{Pl^2}{EI} \dots \dots \dots (12)$$

\* Cf. Rayleigh's 'Theory of Sound,' vol. i. 1894, p. 299.

and when  $P$  is zero (equation (4))

$$\phi = \theta = 2.36502$$

for vibrations of the fundamental type.

Now, when  $Pl^2/EI$  is small,  $\theta$  differs little from 2.36502, and we have, to the first order in this difference

$$\phi = \left[ \phi \right]_{\theta=2.365} + \left[ \frac{d\phi}{d\theta} \right]_{\theta=2.365} (\theta - 2.365),$$

that is

$$\alpha l = 10.5083 - 3.4432 \beta l. \quad . \quad . \quad . \quad (13)$$

Eliminating  $\alpha$  from (12) and (13), we find, after an expansion by the Binomial as far as the second term

$$\beta^2 l^2 = 5.5933 - .2251 \frac{Pl^2}{EI},$$

by means of which the second of (10) gives

$$k^2 = 31.285 \frac{EI}{\rho \omega l^4} \left\{ 1 + .09831 \frac{Pl^2}{EI} - .00557 \left( \frac{Pl^2}{EI} \right)^2 \right\}. \quad (14)$$

§ 12. For harmonics we may in all cases take

$$\tanh \phi = 1;$$

thus, reasoning as in the last paragraph,

$$\alpha l = (1 - \eta) \beta l + \frac{1}{2} \eta^2,$$

where

$$\eta \equiv (2i - \frac{1}{2})\pi,$$

whence

$$\beta^2 l^2 = \frac{1}{4} \eta^2 - \frac{1}{\eta} \frac{Pl^2}{EI}$$

and

$$k^2 = \left( \frac{\eta}{2} \right)^4 \frac{EI}{\rho \omega l^4} + \frac{\eta}{2} \left( \frac{\eta}{2} - 1 \right) \frac{P}{\rho \omega l^2} - \frac{\eta - 1}{\eta^2} \frac{P^2}{\rho \omega EI}. \quad (15)$$

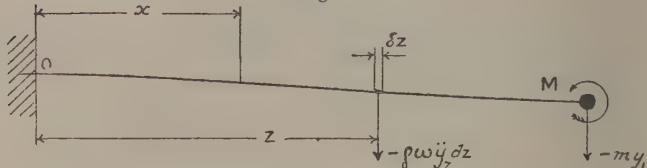
#### Section IV.—Massive Bar, Load at Directed End.

§ 13. When a massive clamped-directed bar carries a load concentrated at its directed end, the frequency and type of lateral vibration can be found to any required degree of accuracy by the method of continuous approximation\*.

\* See Phil. Mag., July 1905, pp. 113-125, and March 1906, pp. 354-374.

In figure 4, let the origin be at the clamped end and

Fig. 4.



assume for the first approximation to the vibration curve

$$y = y_1 \{3(x/l)^2 - 2(x/l)^3\}, \quad \dots \quad (16)$$

which, at the clamped end, satisfies  $y = \frac{dy}{dx} = 0$ , and, at the directed end,  $y = y_1$ ,  $\frac{dy}{dx} = 0$ .

Let  $m$  = the mass concentrated at the end,

$M$  = the couple required to maintain the direction at that end.

Then the ordinary approximate theory leads to

$$EI \frac{d^2 y}{dx^2} = -\rho \omega \frac{\ddot{y}_1}{y_1} \int_x^l y_z (z-x) dz - m \ddot{y}_1 (l-x) + M. \quad (17)$$

Inserting the value of  $y_z$  from (16) and performing the integrations,

$$\begin{aligned} -EI y = \rho \omega \ddot{y}_1 \left( \frac{13}{120} l^2 x^2 - \frac{1}{12} l x^3 + \frac{1}{120} \frac{x^6}{l^2} - \frac{1}{420} \frac{x^7}{l^3} \right) \\ + m \ddot{y}_1 \left( \frac{l x^2}{4} - \frac{x^3}{6} \right), \quad \dots \quad (18) \end{aligned}$$

the constants of integration and the value of  $M$  having been determined by the end conditions. This expression is the second approximation to the vibration type, and is often sufficiently accurate.

Putting  $x = l$  we have

$$k^2 = \frac{EI}{\frac{13}{420} \rho \omega l^4 + \frac{1}{12} m l^3}. \quad \dots \quad (19)$$

§ 14. Proceeding to a still closer approximation we can insert in (17) the value of  $y_z$  given by (18) and again

integrate. In the resulting expression for  $y$ , we can again put  $x=l$  and get

$$k^2 = \frac{EI}{\left( \frac{.3832 \rho \omega l + m}{.3714 \rho \omega l + m} \right) \frac{13}{420} \rho \omega l^4 + \frac{ml^3}{12}} \cdot \cdot \cdot \quad (20)$$

The portion shown in brackets is the correction which this result gives to that first obtained. When the mass of the load is equal to that of the bar, (20) reduces to

$$k^2 = \frac{8.730 EI}{\rho \omega l^4}.$$

#### Section V.—Loaded Bars; Mass of Bar Neglected.

§ 15. When the mass of the bar is neglected and vibrations occur under the action of a concentrated load only, the exact solutions can be determined directly.

If there be no longitudinal force and the mass  $m$  be at the directed end, the equation of equilibrium is

$$EI \frac{d^2 y}{dx^2} = -m \ddot{y}_1 (l - x) - M.$$

$$\therefore EI y = -m \ddot{y}_1 \left( \frac{1}{4} l x^2 - \frac{1}{6} x^3 \right),$$

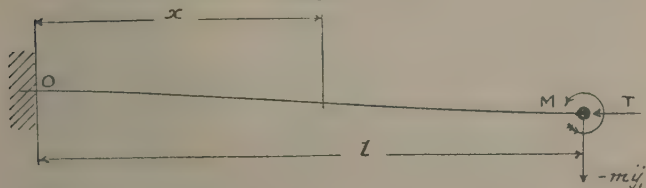
and

$$k^2 = \frac{12 EI}{ml^3} \cdot \cdot \cdot \cdot \cdot \cdot \quad (21)$$

§ 16. *Longitudinal Thrust*.—If in addition to the load  $m$  there be a longitudinal thrust  $T$  acting on the massless bar as represented in figure 5,

$$EI \frac{d^2 y}{dx^2} = -m \ddot{y}_1 (l - x) + T(y_1 - y) - M.$$

Fig. 5.



By virtue of the conditions that at  $x=0$ ,  $y = \frac{dy}{dx} = 0$ ,

and, at  $x = l$ ,  $\frac{dy}{dx} = 0$ , the solution may be written

$$y = \frac{m\ddot{y}_1}{nT} \left\{ \frac{\cos nl - 1}{\sin nl} (1 - \cos nx) + nx - \sin nx \right\}$$

in which  $n^2 = T/EI$ ; and putting  $x = l$ ,

$$k^2 = \frac{T}{ml} \frac{nl \sin nl}{2 - 2 \cos nl - nl \sin nl} \quad (22)$$

This result might have been deduced from that for a bar clamped at each end with a load at the centre\*.

Equation (22) shows that  $k^2 = 0$  and vibration ceases when  $nl = \pi$ , that is, when

$$T = EI \pi^2/l^2.$$

After expansion, (22) becomes

$$k^2 = \frac{12EI}{ml^3} \left( 1 - \frac{4}{15} n^2 l^2 + \dots \right),$$

agreeing with (21) when  $T$ , and therefore  $n$ , is zero.

When  $nl = 2i\pi$  ( $i$  being an integer) the right-hand side of (22) is indeterminate but has the limit  $-T/ml$ , vibration being impossible if  $T$  is positive.

§ 17. The result may also be written

$$k^2 = \frac{T}{ml} \frac{\frac{1}{2} nl}{\tan \frac{1}{2} nl - \frac{1}{2} nl},$$

from which ( $nl$  being positive) the expression for the frequency is real when  $\tan \frac{1}{2} nl > \frac{1}{2} nl$ , that is, for values of  $nl$  between 0 and  $\pi$ , and for decreasing intervals in the neighbourhood of  $3\pi$ ,  $5\pi$ , &c.; vibration always ceasing when  $nl = (2i-1)\pi$ .

The first of these intervals is from  $nl = 8.9868 \dots$  to  $nl = 3\pi$ , and this corresponds to the first harmonic.

With the massless bar harmonics are impossible with low values of  $T$ . As  $T$  is increased the frequency falls, becoming zero when  $nl = \pi$ ,  $\ddot{y}$  being then zero. Between  $\pi$  and  $8.9868$  vibration is impossible, the deflexion increasing with the time. If this region be safely passed that between  $8.9868$  and  $3\pi$  is reached, during which vibration may again occur,

\* Phil. Mag. September 1906, p. 243.



T being now sufficient to enable the bar to assume the curve of the first harmonic type.

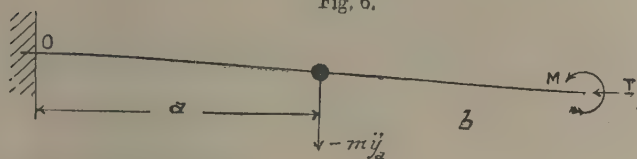
§ 18. If the mass be situated at any point in the length, let it divide the bar into segments  $a$  and  $b$ , as indicated in fig. 6. Then for  $x < a$  the differential equation becomes

$$EI \frac{d^2 y}{dx^2} = -m \ddot{y}_a (a - x) + T(y_1 - y) - M,$$

of which the solution is

$$y = A_1 \sin nx + B_1 \cos nx + \frac{m \ddot{y}_a}{T} (x - a) + y_1 - \frac{M}{T}.$$

Fig. 6.



For  $x > a$  the equation is

$$EI \frac{d^2 y'}{dx^2} = T(y_1 - y') - M,$$

and the solution

$$y' = A_2 \sin nx + B_2 \cos nx + y_1 - \frac{M}{T},$$

where the dashed symbols refer to values of  $y$  in the region  $b$ .

The conditions at  $x=0$  and  $x=l$  lead respectively to

$$\left. \begin{aligned} y &= -\frac{c}{n^2} \sin nx + \left( \frac{m \ddot{y}_a}{T} a + \frac{M}{T} - y_1 \right) (\cos nx - 1) + \frac{m \ddot{y}_a}{T} x \\ y' &= \frac{M}{T} (\sin nl \sin nx + \cos nl \cos nx) + y_1 - \frac{M}{T} \end{aligned} \right\} \quad (23)$$

The conditions of continuity of  $y$  and  $\frac{dy}{dx}$  when  $x=a$  give an expression for the couple at the directed end, namely

$$M = m \ddot{y}_a \frac{\cos na - 1}{n \sin nl}.$$

Putting  $x=a$  in the expressions for  $y$  and  $y'$  we find

$$k^{-2} = \frac{m}{nT} \left\{ \sin na (2 - \cos na) - \cot nl (1 - \cos na)^2 - na \right\}. \quad (24)$$

When  $a=l$ , this reduces to the result of § 16.

It can be shown, from equations (23), that the amplitude is a maximum at the directed end.

Section VI.—*Deduction of the Results of some Statical Problems.*

§ 19. The calculations in Section V. are similar to those required for the corresponding statical problems. Thus if a clamped-directed bar be subjected to a force  $W$  at and perpendicular to the directed end, the centre-line assumes the form (see § 15)

$$y = \frac{W}{EI} \left( \frac{1}{4} lx^2 - \frac{1}{6} x^3 \right).$$

Similar transformations may be made in §§ 16 and 18, and important results obtained. In each case the maximum deflexion is at the directed end.

When the bar is subjected to a force  $w$  per unit length acting normally to the  $x$ -axis, its deflexion curve is

$$y = \frac{w}{EI} \left( \frac{1}{6} l^2 x^2 - \frac{1}{6} lx^3 + \frac{1}{24} x^4 \right).$$

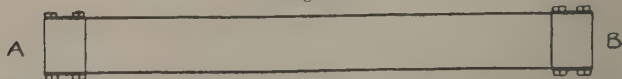
We notice that, for a force concentrated at the end, the maximum deflexion of a clamped-directed bar is one-quarter of that for a clamped-free bar; whilst when the force is uniformly distributed the ratio is one-third.

Section VII.—*Some Experimental Results.*

§ 20. The experiments described in this section were made merely to compare, in a few cases, the frequency calculated by the methods given in this paper with that obtained by direct observation. Care was taken that the accuracy should reach certain limits.

The composite beams used for the experiments consisted of two flat wrought-iron bars, each 3.00 cms. broad by 0.315 cm. thick, the latter dimension being in the plane of

Fig. 7.



vibration. The bars were rigidly connected together at their ends by being securely screwed to massive cast-iron blocks,

as shown in fig. 7. The effective length was measured between the blocks, and this, as well as the distance between the bars, was varied.

A striking feature of the apparatus is the certainty with which the end conditions can be relied upon. When this is the case, the distance between the bars is immaterial; some of the experiments may, in this respect, be taken as a verification of the absolute rigidity of the ends.

§ 21. The flexural rigidity of a piece of one of the bars was determined by supporting it on knife-edges at each end of a span of 58.42 cms. (23 inches), and applying loads at the centre. The observed deflexions are given in the following table.

Load in lbs.	Deflexion. Inches.	Difference. Inches.
0 .....	.627	.047
1 .....	.580	.048
2 .....	.532	.049
3 .....	.483	.050
4 .....	.433	.050
3 .....	.483	.048
2 .....	.531	.049
1 .....	.580	.049
0 .....	.629	
Average difference per pound		.0488

The average deflexion at the centre was thus found to be 0.0488 inch per pound, corresponding to a flexural rigidity of  $15.20 \times 10^6$  grammes weight-cms.<sup>2</sup> The value of  $E$  here involved is the static modulus, but the difference between it and the kinetic modulus would be too small to affect the calculations.

§ 22. The results of the experiments on the compound bars are given in the following table. In each case vibrations took place in a horizontal plane, that is, in the plane of fig. 7.

The clamped end was very securely bolted to a large lathe-bed in such a manner as to leave no room for doubt as to the applicability of the terminal conditions at that end.

No. of Experiment.	Effective Length. cms.	$\frac{1}{2}$ Weight at Directed end. gms.	Distance apart of bars. cms.	Observed Frequency.	Calculated Frequency.
1 .....	114.3	1604	16.5	1.27	1.26
2 .....	114.3	780	16.5	1.66	1.67
3 .....	91.5	1604	16.5	1.78	1.78
4 .....	91.5	780	16.5	2.40	2.40
5 .....	114.3	1604	6.0	1.26	1.26
6 .....	114.3	802	6.0	1.65	1.65

The last column is calculated from equation (20); the values are, however, practically identical with those given by (19). It will be seen that the agreement between the observed and calculated frequencies is very satisfactory.

I should like to take this opportunity of making a slight correction in my paper "On the Lateral Vibration of Bars subjected to Forces in the Direction of their Axes." On page 236 of the Philosophical Magazine for September 1906, and on page 227 of the Proceedings of the Physical Society, vol. xx., instead of  $\tanh \phi$  I have used its reciprocal. The error makes very little difference to the final equation, the correct expression being

$$-\frac{\ddot{y}_1}{y_1} = 500.56 \frac{EI}{\rho \omega l^4} + 12.91 \frac{P}{\rho \omega l^2} - .167 \frac{P^2}{\rho \omega EI}.$$

Armstrong College,  
January 1908.

XXXV. *An Experimental Investigation of Gibbs's Theory of Surface-Concentration, regarded as the basis of Adsorption.*

(Second Paper.) By WM. C. McC. LEWIS, M.A.\*

[From the Muspratt Laboratory of Physical and Electro-chemistry, University of Liverpool.]

IN a previous paper† upon this subject a number of determinations were recorded, which had been carried out with a view to investigate experimentally the expression deduced by Gibbs for the surface condensation of one of the components of a two-phase system at the boundary separating the phases. Two immiscible liquids—namely a hydrocarbon oil and distilled water—were brought into contact, there being a definite interfacial tension existing at the boundary of separation. To the water there were added various substances soluble in water, but insoluble in oil, viz. sodium glycocholate, methyl-orange, and Congo-red. These substances all produced a lowering effect on the interfacial tension—such lowering effect increasing with corresponding increase in concentration of the solute.

Gibbs's expression connecting the surface-concentrating of the solute with the lowering effect on the tension and the concentration of the solute in the bulk of the (aqueous) phase is

$$\Gamma = -\frac{c}{RT} \frac{d\sigma}{dc}$$

where  $\Gamma$  = the mass (in grams) of the solute per square centimetre of the dividing surface in excess of what would be there supposing the concentration to remain the same as in the bulk of the solution ;

$c$  = the bulk concentration of the solute in the aqueous phase ;

$R$  = the gas constant ;

$T$  = the absolute temperature ;

$\sigma$  = the interfacial tension ; and

$\frac{d\sigma}{dc}$  = the rate of change of the tension with the concentration of the solute.

\* Read November 27, 1908.

† Lewis, Phil. Mag. April 1908 ; also Proc. Phys. Soc. vol. xxi. p. 150 (1908).



The negative sign on the right-hand side indicates that  $\frac{d\sigma}{dc}$  must be negative in order that  $\Gamma$  may be positive, *i. e.* those substances which lower the interfacial tension will increase in concentration at the interface.

It was shown how this expression might be applied to the phenomenon of adsorption, which is considered to be more or less a surface condensation. The determinations referred to consisted in measurements of the various quantities occurring in the above expression, and in comparing the values obtained on the two sides of the equation. The general results showed that the actually determined values for  $\Gamma$  far exceeded the calculated values—those found being of the order  $10^{-6}$  grms./cm.<sup>2</sup>, while the calculated ranged from  $10^{-7}$  to  $10^{-8}$  grms./cm.<sup>2</sup> This discrepancy was observed in the case of all the substances mentioned in the previous paper. The adsorption coefficient ( $\Gamma$ ) was measured experimentally under two quite different conditions—namely, at an approximately *plane* surface; and secondly at a very curved surface, with concordant results in both cases.

With the view of throwing some light on the cause of this discrepancy, further determinations of a similar character were carried out and are recorded in the present paper.

It may be pointed out that the substances mentioned are electrolytes (sodium salts of organic acids) and would therefore suffer considerable dissociation in aqueous solution. To account for the discrepancy referred to one possibility suggested itself, namely, the presence of some kind of "double layer" electrical effect. Of course this may or may not be the cause, and it will be shown that for these bodies this is probably *not* the cause, but evidently the first step was to examine other electrolytes—ordinary inorganic salts of simple constitution and comparatively low molecular weight, and compare the effects obtained in these cases with that obtained for sodium glycocholate and the dye-stuffs.

Finally, to carry out adsorption determinations if possible with typical non-electrolytes and compare the results obtained in this case also.

THE INTERFACIAL TENSION BETWEEN OIL AND WATER  
MEASURED BY THE DROP-PIPETTE.

The pipette was identical with that shown in fig. 1 (previous paper). An expression giving the tension directly in absolute units may be derived as follows:—

Consider the equilibrium of a drop of oil formed in water and just about to break away from the nozzle of the pipette.

- Let  $\sigma$  = the interfacial tension ;  
 $a$  = the radius of the orifice ;  
 $r$  = the mean radius of the drop ;  
 $\rho_w$  = the density of water ;  
 $\rho_0$  = the density of oil.

The forces tending to draw the drop off are (1) gravitational, due to the difference in densities of the oil and water

$$= \frac{4}{3} \pi r^3 (\rho_w - \rho_0),$$

and (2) the difference of pressure due to curvature,

$$= 2\pi a^2 \frac{\sigma}{r}.$$

The opposing force is due to the tension and  $= 2\pi a \sigma$ .

Hence

$$2\pi a \sigma - 2\pi a^2 \frac{\sigma}{r} = \frac{4}{3} \pi r^3 (\rho_w - \rho_0).$$

Applying this to the particular case of the oil in which

$$\begin{aligned} \rho_w &= 1.0, \\ \rho_0 &= 0.899, \\ a &= 0.175 \text{ cm.}, \\ r &= 0.4125 \text{ cm.}, \end{aligned}$$

we obtain  $\sigma = 0.0467$  gram per cm.

or 45.8 dynes per cm.

Employing the empiric formula of Lohnstein \* and Kohlrausch † the value 40 dynes per cm. is obtained.

The above expression had been already obtained when it was found that an almost identical one had been deduced by

\* Lohnstein. See previous paper.

† Kohlrausch. See previous paper.

G. Guglielmo \* for liquid-air surfaces. He has applied it to the case of the water-air tension with the following results :—

Radius of orifice of pipette.	Tension in milligrams/mm.
0.6 mm.	7.78
1.1 „	7.60
2.15 „	7.69

} mean 7.69 at 25° C.

The value obtained by the capillary-tube method is 7.485 at 25° C. The results are therefore in fair agreement.

As in previous determinations, the drop-pipette was employed for the various cases of adsorption dealt with in the present paper, the only difference being that the absolute values for the tension are here calculated by means of the expression deduced above.

#### THE INTERFACIAL TENSION BETWEEN A HYDROCARBON OIL AND AQUEOUS SOLUTION OF ELECTROLYTE.

The oil employed was similar to that previously experimented with. About twenty electrolytes in all were examined and all showed a *lowering* effect on the oil-water tension—in general monovalent ions of small atomic weight having least effect while copper and barium salts had a marked influence. This lowering of the tension is remarkable in view of the fact that inorganic salts (with the exception of lithium) *raise* the air-water tension †. Further, it is a general property of salts to be adsorbed by charcoal; and if this is regarded as indicative of a lowering of tension at the charcoal surface, one may be perhaps justified in saying that for aqueous solutions of an electrolyte at a liquid or solid ‡ interface, one may expect in all cases a lowering of tension accompanied by adsorption, while on the other hand at the solution-air surface no such generalization can be made.

A certain number of these salts were selected for adsorption determinations—in the case of caustic soda the emulsion

\* G. Guglielmo, *Accad. Lincei, Atti*, xv. p. 287 (1906).

† Whatmough, *Zeitschr. Phys. Chem.* vol. xxxix. p. 129 (1901).

‡ Using the word in its ordinary sense.

method\* being employed; while with the rest the large-dropping apparatus was used—this being rendered necessary as the adsorption determinations were carried out for each ion of the salt separately, the concentration changes being estimated by ordinary volumetric and gravimetric means.

*The Adsorption of Caustic Soda on Hydrocarbon Oil.*

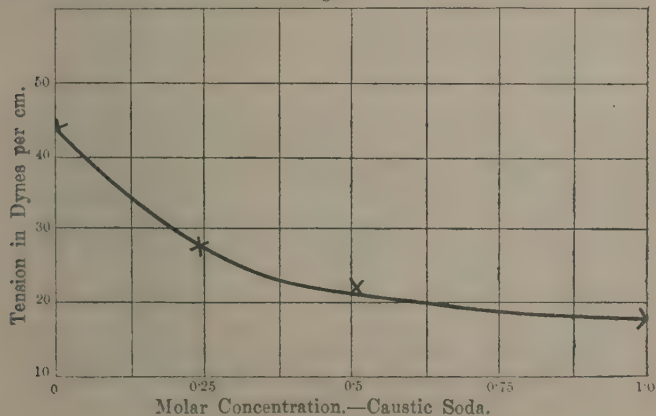
The values of the interfacial tension for different concentrations of the base (corrected for density) are summarized in the following table.

TABLE I.

Concentration		Interfacial Tension in dynes/cm.
per cent.	in gram-moles/litre.	
0	0	45.81
1.0	0.25	26.93
2.0	0.5	22.49
4.0	1.0	16.76
8.0	2.0	streaming from orifice of pipette — tension too small to be measured.

The values of the tension and concentration are plotted in fig. 1.

Fig. 1.



\* See former paper.

*Calculation of  $\Gamma$  by means of Gibbs's expression :—*

Take as a particular case the concentration 0.3 per cent.

$$c = 0.003 \text{ grm./cc.}$$

$$R = \frac{2 \times 4.2 \times 10^7 \text{ ergs}}{\text{molecular weight}} = \frac{2 \times 4.2 \times 10^7}{20},$$

assuming complete dissociation  
at this dilution.

$$T = 288^\circ \text{ abs.}$$

$\frac{d\sigma}{dc}$ , i. e. the tangent to the curve (fig. 1) at the concentration  
0.3 per cent.,

$$= - \frac{9 \text{ dynes/cm.}}{0.003 \text{ grm./c.c.}} = 3000.$$

Hence

$$- \frac{c}{RT} \frac{d\sigma}{dc} = \frac{7.5 \times 10^{-9} \text{ grams per cm.}^2}{\text{or } 1.9 \times 10^{-10} \text{ gram-moles per cm.}^2}$$

It has been assumed here in using the term gram-moles that the substance has been adsorbed either as undissociated base or in ionically equivalent proportions. Gibbs's expression for surface concentration does not contain any term allowing for the preferential adsorption of a single ion if any should occur. Also the measurement of concentration changes by means of changes in tension embodies the same assumption.

*Experimental Determination of  $\Gamma$  :—*

500 c.c. of caustic-soda solution (concentration 0.3 per cent.) were shaken with 1.075 c.c. of the oil until a uniform emulsion resulted. After emulsification a tension determination showed that there had been a decrease in concentration of 0.003 per cent. The diameter of the emulsion particles as measured in the microscope was on the average  $3 \times 10^{-5}$  cm., so that the total adsorbing surface was 107500 cm.<sup>2</sup>

Hence

$$\Gamma = 1.5 \times 10^{-7} \text{ grm./cm.}^2 \text{ or } 3.7 \times 10^{-9} \text{ gram-moles/cm.}^2$$

$$\Gamma \text{ (calculated)} = 7.5 \times 10^{-9} \text{ grm./cm.}^2$$

$$\text{or } 1.9 \times 10^{-10} \text{ gram-moles/cm.}^2$$

The probable error in the experimental result may amount to 25 per cent. A comparison of the observed with calculated values shows once more the considerable discrepancy which exists between them. It should be observed, however, that the absolute value of the experimental determination has fallen from the order  $10^{-6}$  (obtained in previous cases) to  $10^{-7}$  gram./cm.<sup>2</sup> It should also be mentioned that if any slight trace of acid were present in the oil the adsorption of a base would be considerably altered.

*Re-determination of the Adsorption of Sodium Glycocholate.*

Having obtained a purer specimen of the material than that used in former cases, the values for the tension between the oil and solutions of different concentrations were again determined. There were only small changes noted in the result—the calculated value for the adsorption (at a concentration 0.25 per cent.) being  $7 \times 10^{-8}$  gram./cm.<sup>2</sup>, while in the former case the value was about  $5 \times 10^{-8}$ . The actual value determined by experiment was from 3.5 to  $4.7 \times 10^{-6}$  gram./cm.<sup>2</sup> while in the previous case it had lain between the limits 3.1 and 5.4 gram./cm.<sup>2</sup>

*The Adsorption of Silver Nitrate.*

In the measurements of the adsorption of this substance (and in all subsequent cases) the "large drop" \* apparatus was employed. One litre of  $\frac{1}{50}$  molar silver nitrate was prepared and about 750 c.c. poured into the tube through which the oil passed in the form of drops (of about 0.3 cm. diameter), and on the surface of these the solute was adsorbed. When the oil-reservoir (of 1 litre capacity) had been exhausted and the total number of drops estimated from the time taken in emptying the reservoir, the solution which had been thus treated was withdrawn and analysed. For this particular substance the silver only was estimated before and after the experiment by precipitation with hydrochloric acid, the Gooch crucible method being employed. An extremely small difference in weight was observed,

\* See previous paper.



namely 1.6 milligrams of silver chloride. The total surface area of the oil was 7000 cm.<sup>2</sup>

Hence the adsorption of the silver is

$$\Gamma_{Ag} = \underline{1.7 \times 10^{-8} \text{ grm./cm.}^2}$$

A repetition of the determination gave

$$\Gamma_{Ag} = 3.4 \times 10^{-8} \text{ grm./cm.}^2$$

The mean value is therefore

$$\Gamma_{Ag} = \underline{2.5 \times 10^{-8} \text{ grm./cm.}^2}$$

It may be mentioned that the measuring vessels used in these and subsequent determinations were recalibrated for 15° C., and all solutions were cooled to this temperature in a bath before measuring out the volume.

It will be noted that the value obtained above is much smaller than for the previously mentioned substances (except caustic soda). One might therefore expect a closer approximation to the value calculated on Gibbs's expression.

#### *Calculation of the Adsorption of Silver Nitrate.*

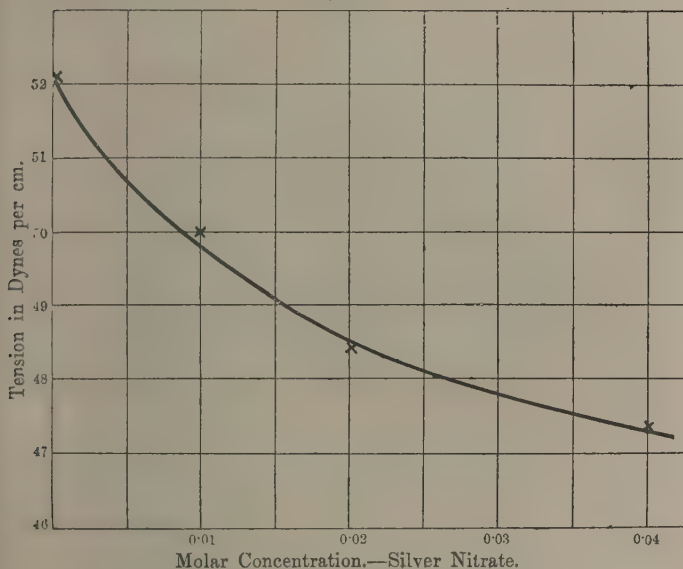
As in previous cases the oil-solution tension curve was obtained by means of the drop-pipette. The oil employed for these measurements was not quite identical with that used with caustic soda. The values obtained (corrected for density) are given in the following table.

TABLE II.  
Oil—Silver-Nitrate Solutions.

Concentration. in gram-moles per litre.	Pipette Drop-number.	Tension dynes, cm.
0	135	52
0.01	140	50
0.02	145	48.4
0.04	149	47.3

The values of the concentration are plotted against tension in fig. 2.

Fig. 2.



The calculated adsorption, namely

$$\frac{c}{RT} \frac{d\sigma}{dc},$$

has the following value for a concentration  $\frac{1}{50}$  molar :—

$$c = 0.0034 \text{ gram. per c.c.}$$

$$R = \frac{8.4 \times 10^7 \text{ ergs}}{\text{molecular weight}}.$$

The molecular weight at this dilution has been found by Smits\* to be 94.  $T = 289^\circ \text{ abs.}$

$\frac{d\sigma}{dc}$  is the tangent to the curve at  $\frac{1}{50}$  molar concentration and is equal to

$$\frac{1.9 \text{ dyne/cm.}}{0.0034 \text{ gram./c.c.}} = 556.$$

Hence

$$\frac{c}{RT} \frac{d\sigma}{dc} = 7.3 \times 10^{-9} \text{ gram per cm.}^2$$

\* Smits, *Zeitschr. Phys. Chem.* vol. xxxix. p. 418 (1902).

This refers to the salt  $\text{AgNO}_3$ , so that on the assumption that each ion is adsorbed equivalently we obtain the calculated value for the silver ion to be

$$\Gamma_{\text{Ag}} = \underline{4.5 \times 10^{-9} \text{ grm./cm.}^2}$$

$$\Gamma_{\text{Ag}} (\text{found}) = \underline{2.5 \times 10^{-8} \text{ grm./cm.}^2}$$

The values for the experimental and calculated adsorption are thus in much closer agreement than in any previous case—the experimental being five times the calculated. The question is: Does this represent a real discrepancy even in this case, or is it to be considered as agreement within the limits of experiment? It is difficult to say, but without laying too great stress upon it, I am of opinion that there is a real (though small) discrepancy between calculated and observed values.

#### *The Adsorption of Potassium Chloride.*

Exactly similar determinations were carried out with this substance as in the case of silver nitrate, except that with this salt both the metal and acid were estimated.

*The Adsorption of the Potassium of Potassium Chloride.*—A very large quantity of oil was allowed to pass through the KCl solution in the large-drop apparatus. The resulting oil-surface area was  $16,000 \text{ cm.}^2$ . The quantity of potassium present in the solution before and after adsorption was estimated as potassium sulphate. The difference in weight of the  $\text{K}_2\text{SO}_4$  precipitate was  $3.2$  milligrams, whence the adsorption of the potassium is

$$\Gamma_{\text{K}} = 8 \times 10^{-8} \text{ grm./cm.}^2$$

A repetition of the experiment gave

$$\Gamma_{\text{K}} = 2 \times 10^{-8} \text{ grm./cm.}^2$$

Thus the mean value is

$$\Gamma_{\text{K}} = \underline{5 \times 10^{-8} \text{ grm./cm.}^2}$$

*Calculation of  $\Gamma_{\text{K}}$  on Gibbs's formula:—*

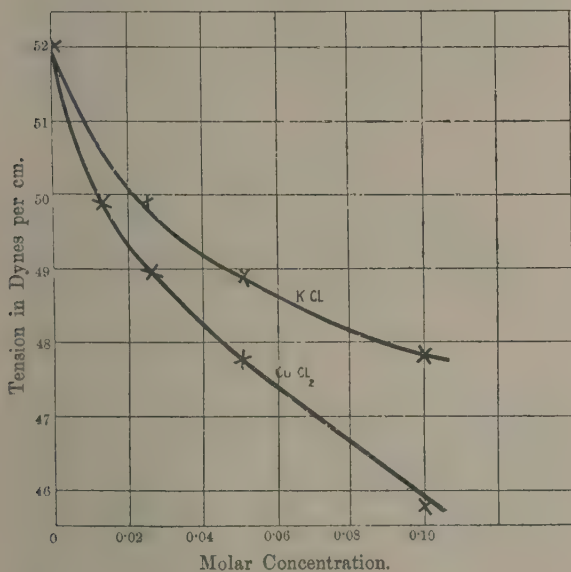
The values of the tension corresponding to different concentration (corrected for density) are given as follows:—

TABLE III.—Hydrocarbon Oil—Potassium Chloride.

Concentration in gram-moles per litre.	Pipette Drop-number.	Tension dynes/cm.
0	135	52
0.0125	139	50.4
0.025	140	49.9
0.05	143	48.9
0.1	147	47.8

The tension-concentration curve is given in fig. 3.

Fig. 3.



As before, the calculated adsorption for  $\frac{M}{20}$  KCl is

$$c = 0.0037 \text{ gram./c.c.}$$

$$R = \frac{8.4 \times 10^7 \text{ ergs.}}{37}$$

where 37 is the osmotic molecular weight.

$$T = 289^\circ \text{ abs.}$$

$$\frac{d\sigma}{dc} = 478.$$

Hence

$$\frac{c}{RT} \frac{d\sigma}{dc} = \underline{2.3 \times 10^{-9} \text{ gram KCl per cm.}^2 \text{ surface,}}$$

which corresponds to  $1.7 \times 10^{-9}$  grm. of potassium,

while  $\Gamma_{\text{K}}$  (found) is  $5 \times 10^{-8}$  grm./cm.<sup>2</sup>

It will be noted that potassium shows a greater discrepancy than silver as regards the calculated value. The results are of about the same order as that obtained for caustic soda.

*Estimation of the Adsorbed Chlorine :—*

The surface area of the oil was 10,000 cm.<sup>2</sup> The change in the chlorine concentration was determined as silver chloride, the difference in weight of AgCl precipitate due to adsorbed chlorine being 0.35 milligram. Hence

$$\Gamma_{\text{Cl}} = 8 \times 10^{-9} \text{ grm./cm.}^2$$

Repetition gave  $\Gamma_{\text{Cl}} = 3 \times 10^{-9} \text{ grm./cm.}^2$

These quantities are, however, so small that one is only justified in stating that the adsorption of the chlorine is of the order  $10^{-9}$  grm./cm.<sup>2</sup>

The calculated value is  $1.6 \times 10^{-9}$  grm./cm.<sup>2</sup>, so that the agreement between calculated and observed is fairly good.

These results for potassium chloride appear to point (at least as far as the small changes in concentration observed can be trusted) to the possibility of a certain amount of separation of the ions in the process of adsorption, *i.e.* a preferential cation adsorption. If this is so, it at once suggests

some electrical effect taking place which is not taken account of in Gibbs's capillary theory.

*The Adsorption of Barium Chloride.*

In the case of this substance the chlorine only was estimated, this being carried out volumetrically with  $\text{AgNO}_3$  solution. The surface area of the oil was  $4000 \text{ cm.}^2$ . In the titration the change in concentration of chlorine due to adsorption was less than  $0.1 \text{ c.c. AgNO}_3$  solution. From this one can only say, therefore, that the adsorption  $\Gamma_{\text{Cl}}$  is not of greater magnitude than  $\underline{10^{-8} \text{ grm./cm.}^2}$ .

*Calculation of the Adsorption :—*

The following table (IV.) contains the values obtained for the tension (corrected for density) at different concentrations.

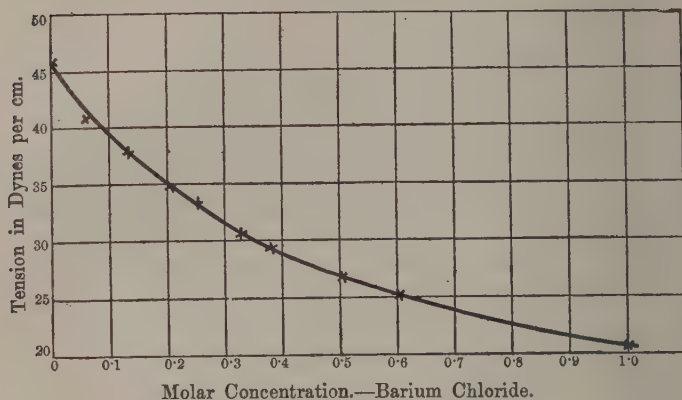
The values of the tension are plotted against concentration in fig. 4. By taking the tangent to this curve the value of  $\frac{d\sigma}{dc}$  is obtained for a particular concentration.

TABLE IV.

Concentration of the Barium Chloride		Density of the Solutions.	Drop-number.	Relative Tension.	Tension in dynes/cm.
Gram-moles per litre.	Per cent. anhydrous.				
0	0	1.000	153	1.000	45.81
0.062	1.29	1.009	171	0.895	40.99
0.125	2.59	1.019	188	0.829	37.98
0.2	4.14	1.038	210	0.757	34.63
0.25	5.18	1.048	222	0.722	33.07
0.33	6.90	1.066	246	0.663	30.37
0.375	7.76	1.073	255	0.644	29.50
0.5	10.35	1.099	290	0.580	26.57
1.0	20.70	1.216	419	0.444	20.34
(1.7) satd. soln.	34.00	1.289	585	0.338	15.48



Fig. 4.



For a solution of strength  $\frac{M}{50}$  we have

$$c = 0.0041 \text{ gram./cc.}$$

$$R = \frac{8.4 \times 10^7}{75},$$

where 75 = osmotic molar weight,

$$T = 289^\circ \text{ abs.}$$

$$\frac{d\sigma}{dc} = 427.$$

Hence

$$\frac{c}{RT} \frac{d\sigma}{dc} = \underline{5.4 \times 10^{-9} \text{ gram BaCl}_2 \text{ per cm.}^2},$$

$$\text{or } \underline{1.6 \times 10^{-9} \text{ gram. chlorine,}}$$

$$\Gamma_{\text{Cl}} (\text{found}) \approx 10^{-8} \text{ gram./cm.}^2$$

The result obtained here is of too qualitative a nature to show whether there is agreement or not.

#### *Adsorption of Copper Chloride.*

With this substance both copper and chlorine were separately estimated.

*The Adsorption of the Copper :—*

The oil-surface area was 8000 cm.<sup>2</sup> The copper in the solution was estimated electrolytically, both before and after the adsorption. The change in weight of the electrode was only 0.2 milligram.

Hence

$$\Gamma_{\text{Cu}} = 2.5 \times 10^{-8} \text{ gm./cm.}^2$$

A repetition of the determination gave

$$\Gamma_{\text{Cu}} = 4.5 \times 10^{-8},$$

or mean value

$$\Gamma_{\text{Cu}} = \underline{3.5 \times 10^{-8} \text{ gram per cm.}^2}$$

*Estimation of the Chlorine :—*

As in previous cases, this was estimated as silver chloride. The difference in weight of the AgCl precipitate was 6 milligrams. The surface-area = 8000 cm.<sup>2</sup>, and hence

$$\Gamma_{\text{Cl}} = 1.8 \times 10^{-8} \text{ gm./cm.}^2$$

A repetition of the determination gave

$$\Gamma_{\text{Cl}} = 2.3 \times 10^{-8};$$

so that the mean value is

$$\Gamma_{\text{Cl}} = \underline{2 \times 10^{-8} \text{ gm./cm.}^2}$$

Hence

$$\Gamma_{\text{Cu}} + \Gamma_{\text{Cl}} = \underline{5.5 \times 10^{-8} \text{ gm./cm.}^2}$$

*Calculation of the Adsorption of CuCl<sub>2</sub> :—*

The following table gives the value of the tension (corrected for density of the solution) at different concentrations :—

Hydrocarbon-Oil and CuCl<sub>2</sub>.

Concentration in gram-moles per litre.	Pipette Drop-number.	Tension in dynes/cm.
0	135	52
0.0125	140	49.9
0.025	143	48.9
0.05	147	47.8
0.1	154	45.8

These values are plotted in fig. 3.

For a solution of concentration M/50

$$c = 0.0027 \text{ grm./cm.}^2,$$

$$\frac{d\sigma}{dc} = 518,$$

$$R = \frac{8.4 \times 10^7}{51}.$$

Where 51 is the molecular weight at this concentration as determined by Biltz\*.

$$T = 289 \text{ abs.}$$

Hence

$$\frac{c}{RT} \frac{d\sigma}{dc} = 4.0 \times 10^{-9} \text{ gram CuCl}_2 \text{ per cm.}^2,$$

$$\Gamma_{\text{Cu}} + \Gamma_{\text{Cl}} (\text{found}) = 5.5 \times 10^{-8} \text{ grm./cm.}^2$$

We have here once more inequality between observed and calculated values. It should be noted that it is the experimental value of  $\Gamma$  which is always the greater, and there are indications that the cation is adsorbed in somewhat greater quantity than the chemical equivalent of the anion.

Further determinations with other electrolytes must be undertaken before one could definitely state whether this important fact—the separation of the ions—takes place or not.

Before going on to discuss in more detail, however, the data already given, the experiments carried out with non-electrolytes will first be briefly recorded.

#### THE ADSORPTION OF NON-ELECTROLYTES.

The chief difficulty at the outset with regard to almost all non-electrolytes is their small solubility in water. It is not easy, therefore, to draw a comparison between the tension effects produced by these bodies and the effects produced by electrolytes. Altogether seventeen non-electrolytes were examined as regards their influence on the interfacial tension. Two of these—aniline and p-nitrosodimethylaniline—were

\* Biltz, *Zeit. Phys. Chem.* vol. xl. p. 199 (1901).

found to be soluble in the oil, and to this is probably due their comparatively marked effects on the tension\*.

The following table (p. 532) contains the values obtained from the various substances examined.

The value obtained for cane-sugar is worthy of note. The lowering effect is small, although a more concentrated solution was employed than in the case of any other substance. The result is in agreement with that observed at the air-surface, in which case the presence of cane-sugar has no appreciable effect.

In the case of all these substances (except iodine which was practically without effect) there is a lowering of tension at the interface which on Gibbs's theory must be accompanied by adsorption. The same general rule, therefore, as regards the effect at the oil-surface seems to hold for non-electrolytes as for electrolytes.

### *The Anomalous Behaviour of Saponin.*

Saponin is a non-electrolyte whose aqueous solutions are characterized by foaming, thus pointing to considerable

#### \* *The Tension between two partially miscible phases.*

The Laplace theory of capillarity shows that the work required to form unit area of interface between two liquids A and B is given by the expression

$$\frac{1}{2}(\rho_A - \rho_B)^2 \int_0^\infty z\psi(z)dz = \sigma_{AB},$$

where  $\rho_A$  and  $\rho_B$  = the densities of the liquids A and B respectively and the integral is supposed constant for all substances. This expression leads to the following relationship:—

$$\sqrt{\sigma_{AB}} = \sqrt{\sigma_{BC}} + \sqrt{\sigma_{CA}},$$

which has not been verified by experiment, but shows in a general way what takes place at the surface. Lord Rayleigh has on this hypothesis calculated the effect of making the transition gradual between A and B by the interposition of  $n$  liquids whose densities are in arithmetical series, in which case it is shown that the energy due to surface tension is reduced to  $(n+1)$  of its original value. Hence any diminution in the abruptness will diminish the energy due to surface-tension. The solubility of the aniline in the oil causes the diminution in abruptness, and consequently we find very marked lowering of tension.

TABLE IV.  
Interfacial Tension : oil-solution.

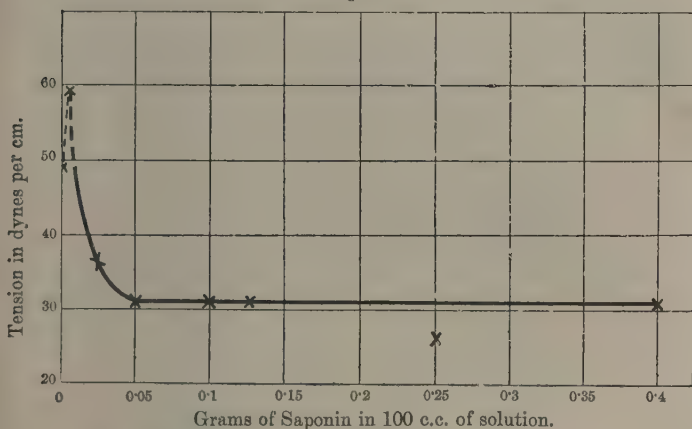
Substance.	Per cent. Concentration.	Molar Concentration.	Relative Tension.
Water .....	.....	.....	1.00
Iodine .....	0.035	$\frac{1}{360}$	slightly > 1.00
Hydroquinone .....	1.0	$\frac{1}{11}$	„ < 1.00
Boric acid .....	0.1	$\frac{1}{62}$	0.97
Urea .....	0.8	$\frac{1}{7}$	0.97
Diphenylthiourea.....	< 0.1	...	0.96
Milk sugar .....	1.0	$\frac{1}{36}$	0.96
Succinimide ..	0.2	$\frac{1}{48}$	0.94
Phenol .....	0.1	$\frac{1}{94}$	0.94
Mercuric cyanide .....	1.0	$\frac{1}{25}$	0.91
p-nitrosodimethylaniline...	< 0.1	$\frac{1}{150}$	0.89
Cane-sugar .....	6.0	$\frac{1}{6}$	0.83
Amygdalin .....	1.0	$\frac{1}{51}$	0.82
Ethyl acetate .....	3.6	$\frac{2}{5}$	0.70
Caffeine .....	1.0	$\frac{1}{21}$	0.71
Aniline .....	3.0	$\frac{1}{31}$	0.63

lowering of tension \*. It has also been noticed that bubbles blown with saponin solution appear to gelatinize, so that when once formed they show a crinkled appearance on contraction. Solutions of this substance showed the following effects at the oil-water interface. On increasing the concentration, there was first of all a rise of tension reaching a maximum at about 0.005 per cent. This was followed by a rapid fall and at 0.025 per cent. a slower fall to 0.05 per cent., beyond which the tension remained constant. Measurements were taken up to 0.4 per cent., at which point streaming

\* See a paper bearing upon this subject by S. A. Shorter, Phil. Mag. vol. xi. p. 317 (1906).

effects began to be observable, so that no further readings could be taken. These peculiarities are shown in fig. 5.

Fig. 5.



It seems probable that the gelatinizing effects which have been observed at the air-surface are also effective at the oil-interface. Experiments were carried out to test whether the substance was soluble in the oil. It was found to be insoluble.

Of the substances enumerated in Table III., it will be seen that caffeine shows the most marked effects in proportion to its concentration. It was therefore selected as a suitable substance wherewith to carry out adsorption determinations.

#### *Determination of $\frac{d\sigma}{dc}$ for Caffeine.*

The material was obtained from Kahlbaum. Its molecular weight was determined by lowering of the freezing-point of water, and was found to be in agreement with the formula (weight 194). The substance employed is the monohydrate and crystallizes from water in long needles. That caffeine is a good example of a non-electrolyte follows from the value for its dissociation constant determined by Wood \*, namely,  $4 \times 10^{-14}$ , which is of the same order as the constant for water.

\* Wood, Journ. Chem. Soc. vol. lxxxix. p. 1844 (1906).



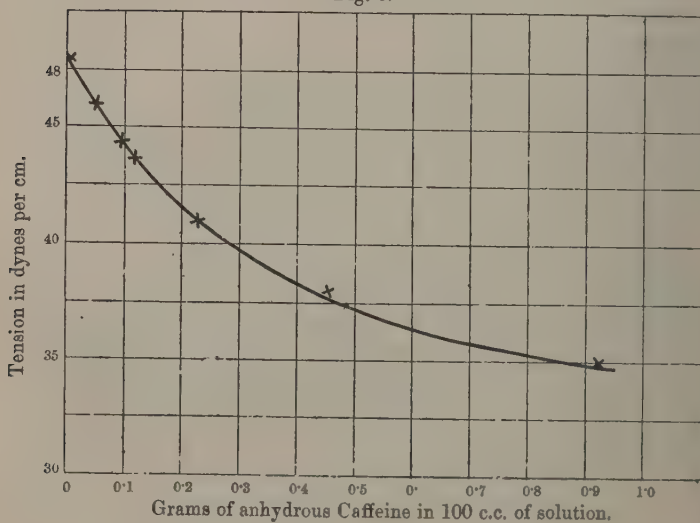
The sample of caffeine was first tested, as in previous cases, for any solubility in the oil. The same drop-number was obtained with the oil filtered from caffeine as for the oil fresh from the stock. Caffeine is therefore insoluble in hydrocarbon oil.

The following table (Table V.) contains the determinations of the interfacial tension for caffeine solutions of different

TABLE V.

Per cent. Concentration of Anhydrous Caffeine.	Drop- number.	Relative Tension.	Tension in dynes/cm.
0	146	1.000	48
0.046	153	0.954	45.8
0.092	158	0.924	44.4
0.114	161	0.907	43.5
0.229	171	0.853	40.9
0.457	184	0.793	38.1
0.915	201	0.726	34.8

Fig. 6.



concentrations, the tension being measured by the drop-pipette. It will be noted that this oil is not quite identical with that previously employed.

The interfacial tension is plotted against concentration, giving the curve on fig. 6. The tangent to the curve gives the value of  $\frac{d\sigma}{dc}$  at the desired concentration.

*Calculation of the Adsorption Coefficient :—*

As before, the expression for the adsorption is

$$\Gamma = -\frac{c}{RT} \frac{d\sigma}{dc}.$$

For a solution of concentration 0.114 per cent. (anhydrous) caffeine the right-hand side of this expression has the following numerical value :—

$$c = 0.114 \text{ per cent.} = 0.00114 \text{ grm./c.c.,}$$

$$R = \frac{2 \times 4.2 \times 10^7}{194} \text{ ergs,}$$

$$T = 288^\circ \text{ abs.,}$$

$$\frac{d\sigma}{dc} = \frac{3 \text{ dynes/cm.}}{0.00114 \text{ grm./c.c.}}$$

$$\therefore \frac{c}{RT} \frac{d\sigma}{dc} = \underline{2.4 \times 10^{-8} \text{ grm./cm.}^2}$$

It will be noted that this value is approximately of the same order as the similarly calculated values for previous substances. The substitution of a non-electrolyte for a simple inorganic salt does not seem therefore to have any marked effect on the theoretical value for the adsorption coefficient. It would seem, therefore, that in both cases we are dealing with a factor of the same magnitude as regards its effect on the tension and the consequent value of  $\frac{d\sigma}{dc}$ . It will be remembered, however, that the results obtained in previous cases pointed to the existence of certain discrepancies between observed and calculated values of  $\Gamma$ . It is of particular interest, therefore, to attempt to measure  $\Gamma$  directly for caffeine, since in this case there is no possibility of separation of ions.

*Determination of  $\Gamma$  for Caffeine in Aqueous Solution.*

The emulsion method\* was employed in this determination.

Solutions of caffeine approximately 0.125 per cent. were prepared, and 500 c.c. shaken in steamed-out glass vessels for three days with about 1-2 c.c. of oil. The resulting emulsions were very uniform under the microscope, the average diameter of the particles being  $6 \times 10^{-5}$  cm. It is worthy of note that this is of the same order as the average diameter of the particles in the previous cases.

The results of two emulsion experiments are given below :—

Caffeine solution (approx. 0.125 per cent.).

Experiment no.	Drop-number before emulsification.	Drop-number after emulsification.	Area of oil-surface.
1 .....	160	160, $160\frac{1}{2}$	87,130 cm. <sup>2</sup>
2 .....	160, $160\frac{1}{2}$	$160\frac{1}{2}$ , $160\frac{1}{2}$	127,310 cm. <sup>2</sup>

The results show that there was *no readable difference in the concentration of the solution before and after emulsification—the difference being a small fraction of a drop*. Special experiments were undertaken to determine the accuracy of the readings made with the pipette, and it was found that for a small drop-number such as 160 the readings were reproducible to within about  $\frac{1}{2}$ th of a drop.

No very reliable quantitative relations can therefore be obtained from these determinations, as we are evidently dealing with a very small change in concentration, *i. e.*, a very small adsorption. The important feature is this—that *the calculated value of  $\Gamma$  (viz.  $10^{-8}$  grm./cm.<sup>2</sup>) would also correspond to a small fraction of a drop as indicating concentration change*; while a value for  $\Gamma$  of the order obtained in the case of sod. glycocholate and the dyes (viz.  $10^{-6}$  grm./cm.<sup>2</sup>) would have been indicated by a change of 4-5 drops, which would of course have been

\* See previous paper.

perfectly readable. These experiments therefore show that *caffeine is very much less adsorbed than sodium glycocholate, Congo red, or methyl orange.*

It is perhaps going beyond the limits of accuracy warranted by the drop-pipette method to make an approximate calculation of the  $\Gamma$  from the data given above. Taking the second experiment, the change in concentration corresponds to about  $\frac{1}{8}$ th of a drop. This is only approximate, and in all probability would give a *maximum* value for  $\Gamma$ . From a curve\* obtained by plotting the drop-numbers against concentrations given in Table IV. it is found that  $\frac{1}{8}$ th of a drop corresponds to a change in concentration of 0.00094 per cent., and hence the total amount of caffeine removed from the 500 c.c. emulsified is 0.0047 gram. The total adsorbing surface is 127310 cm.<sup>2</sup>, hence

$$\Gamma = 3.7 \times 10^{-8} \text{ grm./cm.}^2,$$

$$\Gamma \text{ calculated} = 2.4 \times 10^{-8} \text{ grm./cm.}^2$$

An attempt was at once made to increase the adsorbing area in order to get more trustworthy data. It was found, however, that the limit had almost been reached in the case of the second emulsion—only a little more than 2 c.c. of oil can be emulsified by 500 c.c. of caffeine solution of 0.1 per cent. concentration. The method is therefore limited in this direction. A further attempt at more accurate measurement was made by employing a pipette of much larger oil-capacity (about 500 c.c.). The drop-number against water was 2042, in place of 146 in the case of the smaller pipette. The drop-number against a solution of 0.1 per cent. concentration was 2383. One result was obtained in which the change of concentration before and after emulsification amounted to 5 drops, which corresponded to  $3.5 \times 10^{-8}$  grm. cm., but on attempting to reproduce these values quite variable results were obtained, owing to the change in wetting of the pipette during the long course (over two hours) of a single determination.

It appears probable from the foregoing attempts to measure

\* The curve is a straight line between drop-numbers 155–200. A fraction of a drop can therefore be read fairly accurately.

$\Gamma$  in the case of caffeine, that we are dealing with a quantity too small to be determined with accuracy by the method hitherto pursued. It may be possible by employing, for example, colloidal mercury instead of the oil to determine the caffeine adsorption at a mercury surface with accuracy.

*Behaviour of Sodium Glycocholate and Caffeine in contact with Charcoal.*

In view of the considerable difference between the values for the adsorption of these two substances at the oil-surface, it was of interest to compare the results obtained with charcoal. The material was the same as that employed by Freundlich—namely, Merck's blood-charcoal purified by acid. Freundlich has observed that the further washing of this material does not affect the result of the adsorption determinations, and this was confirmed in the present instance by the fact that there was not the slightest change in the oil-water tension in the case of distilled water which had been shaken up with a quantity of charcoal.

*Adsorption of Sodium Glycocholate:—*

500 c.c. of a solution 0.3 per cent. concentration were treated with one gram of charcoal, and the decrease in bulk-concentration of the solute amounted to 0.125 per cent. The change in concentration was measured by means of the change in value of the oil-solution tension as given by the drop-pipette. The drop-difference (before and after) was 47.

Hence 1 gram charcoal adsorbs from a 0.3 per cent. solution 0.625 gram sodium glycocholate.

*Adsorption of Caffeine:—*

500 c.c. of solution 0.218 per cent. concentration (*i. e.*, approximately equimolar with the sodium glycocholate solution) brought into contact with one gram of charcoal gives a drop-difference of 10, which corresponds on the caffeine curve to a change in concentration of 0.108 per cent.

Hence 1 gram charcoal adsorbs from a 0.218 per cent. solution 0.324 gram of caffeine.

It will be noted that the behaviour of caffeine and sodium glycocholate towards charcoal and oil is very different. In contact with charcoal there appears to be no very marked difference as regards the magnitude of the adsorption for both substances. Possibly this may be due to oxidation of the caffeine by means of the oxygen adsorbed on the charcoal surface.

*Calculation of the Range of the Concentration Effects at the Oil-surface.*

Consider the case of an infinite mass of solution (of a non-electrolyte in water) whose bulk-concentration is  $c_1$  and

surface-concentration  $c_2$ . Suppose one gram-mole of solute transferred from the bulk into the surface-layer. The osmotic work done is

$$RT \log \frac{c_2}{c_1}.$$

There is a corresponding decrease in the surface energy due to adsorption of the solute given by

$$-\frac{d\sigma}{d\Gamma};$$

and equating these two expressions,

$$RT \log \frac{c_2}{c_1} = -\frac{d\sigma}{d\Gamma}.$$

Experimental case:—

If we assume that the solution of caffeine obeys in all probability Gibbs's expression for surface-condensation, from the data previously given we may construct the following table:—

Aqueous solution of Caffeine at 15° C.

Per cent. Bulk-concentration (anhydrous).	Tension in dynes/cm.	$\Gamma$ calculated from Gibbs's equation in $\text{gram./cm.}^2$
0.0	48	0.0
0.046	45.8	$1.2 \times 10^{-8}$
0.092	44.4	1.8 "
0.114	43.5	2.0 "
0.229	40.9	2.3 "
0.457	38.1	2.8 "
0.915	34.8	3.0 "

The tension was plotted against  $\Gamma$ , and the tangent to the curve taken at the point corresponding to a bulk-concentration 0.114 per cent. The tangent gives

$$\frac{d\sigma}{d\Gamma} = \frac{7.25 \text{ ergs/cm.}^2}{1 \times 10^{-10} \text{ gram-moles/cm.}^2} = 7.25 \times 10^{10}.$$

At this point

$$\Gamma = 2 \times 10^{-6} \text{ gram./cm.}^2 = 1 \times 10^{-10} \text{ gram-moles/cm.}^2$$



The equation therefore becomes

$$\log_e c_2 = \log_e c_1 + \frac{7.25 \times 10^{10}}{RT}.$$

$$\therefore \log_{10} c_2 = \log_{10} 0.0064 + 0.044 \\ = -2.1498.$$

$$\therefore c_2 = 0.0071 \text{ gram-moles per litre} \\ = 0.00138 \text{ grm./c.c.}$$

Now if  $D$  is the range through which the surface-concentration effects are practically complete, it follows that the average concentration in the surface-layer is given by the expression.

$$c_2 = \frac{\Gamma}{D},$$

$$\text{or } D = \frac{\Gamma}{c_2};$$

and on substituting the values obtained above it is found that

$$D = 14 \times 10^{-6} \text{ cm.}$$

It is evident that we have here obtained a value for the "range of molecular action," as may be seen by comparing it with other estimates of the same quantity.

		<i>Method.</i>
Quincke .....	$5 \times 10^{-6}$ cm.	"Wedge method."
Reinold & Rücker.	$1.2 \times 10^{-6}$	Limiting thickness of soap film.
Wiener .....	$1.2 \times 10^{-6}$	Effect on phase of reflected light.
Plateau .....	$11.4 \times 10^{-6}$	Limiting thickness of glycerine film.
Drude .....	$1.7 \times 10^{-6}$	" " soap film.
Parks .....	$13.4 \times 10^{-6}$	Wetting of powders.
Johannot .....	from $10^{-6}$ to $10^{-8}$	soap film.

#### DISCUSSION OF RESULTS.

The conclusions to which the foregoing experiments point are:—

- (1) Caffeine in aqueous solution, in all probability obeys Gibbs's law quantitatively.
- (2) Ordinary inorganic salts—Potassium chloride, Silver nitrate, Barium chloride, and Copper chloride—are

adsorbed in quantities which are of the same order as the Gibbs's calculated effect, though the experimentally found values are in all cases *greater* than the calculated. Caustic soda shows a more marked discrepancy between calculated and observed values.

- (3) Complex organic salts—Sodium glycocholate, Sodium oleate\*, Congo red, and Methyl orange—show a very large discrepancy between observed and calculated values. The first thing is therefore to consider the various possibilities of accounting for this large discrepancy.

*The Adsorption of Sodium glycocholate, &c.*

First can the excess exhibited by these substances be explained by an extension of the capillary theory?

The expression

$$\Gamma = - \frac{c}{RT} \frac{d\sigma}{dc}$$

is deduced on the assumption that we are dealing with *one* component only which suffers surface-concentration, *i. e.*, the solute. Since the above-mentioned substances are salts, they will be associated, and almost entirely so, at the dilution worked with. Milner †, in deducing the Gibbs's expression, has allowed for dissociation by the introduction of van 't Hoff's factor (*i*) thus :

$$\Gamma = \frac{c}{iRT} \frac{d\sigma}{dc}$$

This is, however, really the expression employed in the foregoing experiments, since in all cases for the evaluation of *R* (the gas constant) the molecular weight of the solute in *solution* has been employed. By simply making this allowance for dissociation, therefore, the discrepancy between observed and calculated values for adsorption is still very marked—as a matter of fact, such a correction acts in the opposite direction.

The next possible explanation may lie in incorrectness in

\* Reference will be made later to this substance.

† Milner, *Phil. Mag.*, Jan. 1907.

the assumption (which has been made throughout) that the water suffers no surface density-change.

It is very probable that water has not quite the same density in the surface-film as in the bulk of the liquid. Evidence for this is to be found in the phenomenon of heat evolution observed by Parks and others, when water is poured upon finely-divided powders of silica, quartz, glass, &c., where chemical action is of course excluded. Also one must not overlook the fact that in the adsorption of sodium glycocholate and the dyes we are dealing with surface-concentrations far exceeding the ordinarily accepted values for their solubility. Of course in using the term "solubility" we assume that the surface-layer water possesses the same properties as water in bulk. For a solution of sodium glycocholate of 0.25 per cent. bulk concentration,  $\Gamma$  was found to be  $5 \times 10^{-6}$  gm./cm.<sup>2</sup> The value of the range through which the surface-concentration is sensibly different from the bulk has been shown to be approximately  $14 \times 10^{-6}$  cm. Hence the *average* surface-concentration is 0.37 gram per c.c. or 37 per cent., while the ordinarily accepted value for the solubility of sodium glycocholate is 3.9 per cent. There was, however, no visible colour change on the oil surface due to any precipitation.

Again, methyl orange\* showed an adsorption for which  $\Gamma = 5.5 \times 10^{-6}$ , whence the average concentration is 39 per cent. The actual solubility is only 0.078 per cent.

The substances therefore which have shown very great discrepancies as regards Gibbs's theory are *those whose surface concentration greatly exceeds their solubility in the solvent*. For the other substances examined, viz. caffeine and the inorganic salts, *in no case is the ordinary solubility exceeded*, and the adsorption of these substances is in much closer agreement with theory. Thus the surface-concentration of caffeine was found to be 0.26 per cent., while its solubility at ordinary temperature is 1.35 per cent. For silver nitrate the surface-concentration was approximately 0.28 per cent., while its solubility is 12.2 per cent. at 0° C. Similarly for potassium chloride, barium chloride, and copper chloride.

\* See previous paper, *l. c.*

Another possibility however presents itself—namely, the existence of some electrical effect, such as an electrostatic attraction, causing the discrepancy, since sodium glycocholate and the dyes are electrolytes. If this were so, one would also expect somewhat of the same order of discrepancy manifesting itself in the case of the ordinary inorganic salts. Such does not take place. Caustic soda alone has marked effects, and shows an adsorption 20 times greater than the calculated. The other substances generally show a discrepancy of about 5 to 8 times. To whatever cause this smaller discrepancy is due, it is unlikely that the same reason is to be assigned to the very much larger discrepancies observed for sodium glycocholate.

The only other possible means, apparently, of explaining the want of agreement is on the assumption that we are dealing with some irreversible phenomenon of the nature of gelatinization upon the oil surface. The fact of the surface-concentration being much greater than the ordinarily accepted value for the solubility, is in agreement with this view. It should be pointed out, however, that the overstepping of the solubility limit does not seem to be essential to gelatinization, as will be seen by referring to the determinations carried out with saponin. The solubility of this substance is great; according to Beilstein it is completely miscible with water, and although this cannot actually be the case, the constancy of the tension at comparatively dilute solutions cannot be explained by the solubility having been reached. This is a very striking case showing the sensitiveness of the tension to gelatinizing effects. The hypothesis of gelatinization has been put forward by Milner\* to explain the behaviour of sodium oleate, although in this case the tension curve was continuous and quite similar to the sodium-glycocholate curve. From some unpublished experiments by Mr. H. E. Potts in this Laboratory, it was found that sodium oleate showed an adsorption at the oil surface, the magnitude of which was of the order  $10^{-6}$  gm./cm.<sup>2</sup>, while the calculated was of the order  $10^{-8}$  gm./cm.<sup>2</sup>, that is to say a discrepancy very similar to sodium glycocholate.

\* *Loc. cit.*

Mention may also be made that Freundlich and Losev\* have found that certain dyestuffs produced amorphous precipitates on the charcoal surface during adsorption determinations. This was an irreversible process (the precipitates being insoluble in water), and as such might be considered as analogous to gelatinization.

It is possible, therefore, that some irreversible effect of the nature of gelatinization is the real cause of the discrepancy between observed and calculated values in the adsorption of sodium glycocholate, Congo red, and methyl-orange.

*The Adsorption of Inorganic Salts from Aqueous Solution.*

It has already been pointed out that for certain purely inorganic salts, discrepancies are found between observed and calculated values of the adsorption, but that the existence of these discrepancies might be considered as not quite proved owing to possibility of great experimental error. On the whole, however, it is believed that there really does exist a small discrepancy which requires explanation. Attention might be drawn to the fact that in the cases measured the metal, *i. e.* the cation, shows an excess, while in the case of potassium chloride, and possibly also in barium chloride, the chlorine (the anion) is much nearer to the calculated value. This at once suggests some separation of ions, that is partial selective adsorption due to some electrical effect. The value, however, for the adsorption of the chlorine in copper chloride is scarcely in accord with the preceding statement. Granting that there is a tendency for selective ionic adsorption, it is evident that the phenomenon cannot be completely covered by Gibbs's capillary expression, in the deduction of which no allowance is made for any such ionic separation. Of course for non-electrolytes, where no ionic separation is possible, Gibbs's expression ought to hold quite readily, and experiment confirms this, although the determinations are unfortunately of a more qualitative character than one could wish. In electrolytes, the separation of cation from anion may possibly be regarded as

\* Freundlich and Losev, *Zeit. Phys. Chem.* vol. lix. p. 284 (1907).

local electrolysis at the surface of the oil—that in fact we are dealing with a cell of small dimensions the poles of which are the two homogeneous masses, and the heterogeneous layer separating the masses functions as the “electrolytic medium.” If this is so, we must treat the question as one of electrolytic conduction, but first of all it must be shown that there exists an electrical potential-difference between the electrodes, *i. e.* between the oil and the water. Owing to the non-conducting nature of the oil one cannot employ the usual methods, but by emulsifying the oil (*i. e.* converting it into small particles suspended in water) and placing the emulsion in an electrostatic field, it can be observed whether any motion of the particles takes place or not. This was done experimentally as follows:—

A simple Nernst form of U-tube, identical with that employed by Burton\*, having a vertical capillary tube affixed at the bend through which water slowly runs so as to form a definite surface of separation (in both limbs) between the water and the oil-water emulsion, is fitted with platinum electrodes dipping into the emulsion. A potential-difference of 230 volts was maintained between the electrodes, and the movement of the emulsion-water boundary down in one limb and up in the other was read off on a scale attached to the vessel. It was found that the particles moved towards the anode, being therefore negatively charged. The velocity of the particles was  $4.3 \times 10^{-4}$  cm./sec. under a gradient of 1 volt/cm.

Burton† has deduced an expression for the potential existing between the particles of colloidal metals and the water in which they are suspended, viz.:

$$V = \frac{4\pi \eta v}{K \bar{X}},$$

where  $V$  is the p.d. required (in electrostatic units);  
 $K$  is the specific inductive capacity of the medium;

\* Burton, Phil. Mag. ii. p. 434 (1906).

† Burton, *loc. cit.*



$\eta$  is the viscosity of the medium ;

$\frac{v}{X}$  is the velocity of the particles under (electrostatic) unit potential gradient,  
 $X$  being the p.d. maintained between the poles.

Applying this to the present case of oil suspended in water, we obtain

$$V = 0.00048 \text{ electrostatic unit,} \\ \text{or } \underline{0.15 \text{ volt.}}$$

There is thus quite a measurable potential existing between the oil and water ; and in considering the adsorption of electrolytes we must not overlook this fact. It may be mentioned, in passing, that the above value for the potential is quite of the same order as that obtained by Burton for various colloidal metals.

Since the oil has been shown to be negatively charged, one would not be surprised, speaking generally, that the cation should be selectively adsorbed.

#### SUMMARY.

- (1) The adsorption of various substances on a hydrocarbon-oil surface has been measured. The results are collected in the following table :—

Substance.	Adsorption per cm. <sup>2</sup> of the substance assumed to be in the form of <i>undissociated salt</i> or in chemically equivalent ionic proportions.	
	Found value.	Calculated value.
Sodium glycocholate. .	$5 \times 10^{-6}$ grm./cm. <sup>2</sup>	$7 \times 10^{-8}$ grm./cm. <sup>2</sup>
Congo-red .....	$3.7 \times 10^{-6}$	$1.1 \times 10^{-7}$
Methyl-orange .....	$5.5 \times 10^{-6}$	$1.2 \times 10^{-7}$
Sodium oleate .....	$10^{-6}$	$10^{-8}$
Caustic soda .....	$1.5 \times 10^{-7}$	$7.5 \times 10^{-9}$
Caffeine .....	$3.7 \times 10^{-8}$	$2.4 \times 10^{-8}$

Substance.	Adsorption of cation per cm. <sup>2</sup>		Adsorption of anion per cm. <sup>2</sup>	
	Found value.	Calculated value.	Found value.	Calculated value.
Silver nitrate .....	gm./cm. <sup>2</sup> $2.5 \times 10^{-8}$	gm./cm. <sup>2</sup> $4.5 \times 10^{-9}$	gm./cm. <sup>2</sup> .....	$2.3 \times 10^{-9}$
Potassium chloride.	$5 \times 10^{-8}$	$1.7 \times 10^{-9}$	$10^{-9}$	$1.6 \times 10^{-9}$
Barium chloride ...	.....	$3 \times 10^{-9}$	not $> 10^{-8}$	$1.6 \times 10^{-9}$
Copper chloride ...	$3.5 \times 10^{-8}$	$2 \times 10^{-9}$	$2 \times 10^{-8}$	$2 \times 10^{-9}$

- (2) The large discrepancy between observed and calculated values for the first four substances is possibly due to gelatinization upon the oil-surface.
- (3) Caffeine obeys Gibbs's law within the limits of experimental error.
- (4) There is evidence of slight *selective* adsorption, the cation more than the anion, probably due to electrical effects, since the oil is shown to be negatively charged, the potential-difference between the oil and water being approximately 0.15 volt.

In conclusion, I gladly take this opportunity of expressing my gratitude to Professor F. G. Donnan for his advice and assistance in this work.

XXXVI. *On the Elliptic Polarization produced by the Direct Transmission of a Plane Polarized Stream through a Plate of Quartz, cut in a Direction oblique to the Optic Axis, with a Method of Determining the Error of a Plate supposed to be Perpendicular to the Axis.* By JAMES WALKER, M.A., Oxford\*.

1. LET the primitive stream with polarization-vector  $\text{Exp}(i\omega t)$  be polarized in an azimuth  $\alpha$  with respect to the principal section of the quartz plate. This may be replaced by the elliptically polarized stream represented by the vector

$$(\xi_1, \eta_1) = c_1(\cos \beta, -i \sin \beta) \text{Exp}\{i(\omega t + \epsilon_1)\},$$

with its plane of maximum polarization ( $\beta$  being numerically less than  $\pi/4$  and positive or negative according as the quartz is left- or right-handed) in the principal section, together with the oppositely polarized stream defined by the vector

$$(\xi_2, \eta_2) = c_2(-\sin \beta, -i \cos \beta) \text{Exp}\{i(\omega t + \epsilon_2)\},$$

where

$$c_1 \text{Exp}(i\epsilon_1) = \cos \alpha \cos \beta + i \sin \alpha \sin \beta,$$

$$c_2 \text{Exp}(i\epsilon_2) = -\cos \alpha \sin \beta + i \sin \alpha \cos \beta.$$

These oppositely polarized streams traverse the plate with different speeds, and on emergence the second is retarded in phase relatively to the first by an amount  $\delta$ , so that the emergent light will be in general elliptically polarized and may be denoted by

$$(u, v) = c(\cos \gamma, -i \sin \gamma) \text{Exp}\{i(\omega t + \epsilon)\},$$

where

$$c(\cos \gamma \cos \theta + i \sin \gamma \sin \theta) \text{Exp}\{i(\epsilon + \delta/2)\}$$

$$= c_1 \cos \beta \text{Exp}\{i(\epsilon_1 + \delta/2)\} - c_2 \sin \beta \text{Exp}\{i(\epsilon_2 - \delta/2)\},$$

$$c(\cos \gamma \sin \theta - i \sin \gamma \cos \theta) \text{Exp}\{i(\epsilon + \delta/2)\}$$

$$= -ic_1 \sin \beta \text{Exp}\{i(\epsilon_1 + \delta/2)\} - ic_2 \cos \beta \text{Exp}\{i(\epsilon_2 - \delta/2)\},$$

$\theta$  being the angle between the principal section of the plate and the plane of maximum or minimum polarization of the resultant stream, according as  $\gamma$  is numerically less or greater than  $\pi/4$ .

\* Read November 27, 1908.

Substituting for  $c_1 \text{Exp}(\iota\epsilon_1)$  and  $c_2 \text{Exp}(\iota\epsilon_2)$  and writing

$$\tan R = \sin 2\beta \tan \delta/2, \quad \tan \Delta/2 = \cot 2\beta \sin R, \quad . \quad (1)$$

we obtain

$$\begin{aligned} c(\cos \gamma \cos \theta + \iota \sin \gamma \sin \theta) \text{Exp}\{\iota(\epsilon + \delta/2)\} \\ = \cos(\alpha + R) \cos \Delta/2 + \iota \cos \alpha \sin \Delta/2, \\ c(\cos \gamma \sin \theta - \iota \sin \gamma \cos \theta) \text{Exp}\{\iota(\epsilon + \delta/2)\} \\ = \sin(\alpha + R) \cos \Delta/2 - \iota \sin \alpha \sin \Delta/2, \end{aligned}$$

or

$$\begin{aligned} c \left\{ \cos \gamma \cos \left( \theta - \frac{R}{2} \right) + \iota \sin \gamma \sin \left( \theta - \frac{R}{2} \right) \right\} \text{Exp} \left\{ \iota \left( \epsilon + \frac{\delta}{2} \right) \right\} \\ = \cos \left( \alpha + \frac{R}{2} \right) \left( \cos \frac{\Delta}{2} + \iota \sin \frac{\Delta}{2} \right), \\ c \left\{ \cos \gamma \sin \left( \theta - \frac{R}{2} \right) - \iota \sin \gamma \cos \left( \theta - \frac{R}{2} \right) \right\} \text{Exp} \left\{ \iota \left( \epsilon + \frac{\delta}{2} \right) \right\} \\ = \sin \left( \alpha + \frac{R}{2} \right) \left( \cos \frac{\Delta}{2} - \iota \sin \frac{\Delta}{2} \right); \end{aligned}$$

whence

$$\begin{aligned} \frac{\cos \gamma \cos \left( \theta - \frac{R}{2} \right) + \iota \sin \gamma \sin \left( \theta - \frac{R}{2} \right)}{\cos \gamma \sin \left( \theta - \frac{R}{2} \right) - \iota \sin \gamma \cos \left( \theta - \frac{R}{2} \right)} \\ = \cot \left( \alpha + \frac{R}{2} \right) (\cos \Delta + \iota \sin \Delta), \end{aligned}$$

which gives

$$\begin{aligned} \frac{\cos 2\gamma \sin (2\theta - R)}{1 - \cos 2\gamma \cos (2\theta - R)} = \cot \left( \alpha + \frac{R}{2} \right) \cos \Delta \left. \vphantom{\frac{\cos 2\gamma \sin (2\theta - R)}{1 - \cos 2\gamma \cos (2\theta - R)}} \right\} \\ \frac{\sin 2\gamma}{1 - \cos 2\gamma \cos (2\theta - R)} = \cot \left( \alpha + \frac{R}{2} \right) \sin \Delta \left. \vphantom{\frac{\sin 2\gamma}{1 - \cos 2\gamma \cos (2\theta - R)}} \right\} . \quad . \quad (2) \end{aligned}$$

From these we obtain the equations

$$\left. \begin{aligned} \tan (2\theta - R) &= \tan (2\alpha + R) \cos \Delta, \\ \sin 2\gamma &= \sin (2\alpha + R) \sin \Delta, \\ \tan^2 \gamma &= \tan (\alpha + \theta) \tan (\alpha + R - \theta) \end{aligned} \right\} . \quad . \quad (3)$$

that determine the character of the stream emerging from the quartz,  $\beta$  and  $\delta$  and consequently  $R$  and  $\Delta$  being supposed to be known.

2. For the discussion of the elliptic polarization it is more

convenient to express  $\tan 2(\theta - \alpha)$  and  $\sin 2\gamma$  in terms of  $\tan R$ . Solving for  $\tan 2(\theta - \alpha)$  we obtain

$$\begin{aligned}\tan 2(\theta - \alpha) &= \frac{\tan (2\alpha + R) \cos \Delta - \tan (2\alpha - R)}{1 + \tan (2\alpha + R) \tan (2\alpha - R) \cos \Delta} \\ &= \frac{\sin 2R \cos^2 \Delta/2 - \sin 4\alpha \sin^2 \Delta/2}{\cos 2R \cos^2 \Delta/2 + \cos 4\alpha \sin^2 \Delta/2} \\ &= \frac{\sin 2R - \sin 4\alpha \cot^2 2\beta \sin^2 R}{\cos 2R + \cos 4\alpha \cot^2 2\beta \sin^2 R} \\ &= \frac{2 \tan R - 2 \sin 2\alpha \cos 2\alpha \cot^2 2\beta \tan^2 R}{1 + (\cos 4\alpha \cot^2 2\beta - 1) \tan^2 R}.\end{aligned}$$

Also

$$\begin{aligned}\sin 2\gamma &= \sin 2\alpha \cos R \sin \Delta + \cos 2\alpha \sin R \sin \Delta \\ &= \frac{2 \sin 2\alpha \cot 2\beta \tan R + 2 \cos 2\alpha \cot 2\beta \tan^2 R}{1 + (\cot^2 2\beta + 1) \tan^2 R},\end{aligned}$$

since

$$\sin \Delta = \frac{2 \tan \Delta/2}{1 + \tan^2 \Delta/2} = \frac{2 \cot 2\beta \sin R}{1 + \cot^2 2\beta \sin^2 R}.$$

These expressions have the form

$$y = 2(ax + bx^2)/(1 + cx^2),$$

and the maximum and minimum values of  $y$  are  $y = ax_m$  occurring when

$$x = x_m = (b \pm \sqrt{b^2 + a^2 c})/(ac).$$

3. First considering  $\sin 2\gamma$ , we see that  $\sin 2\gamma = 0$ , or the light is plane polarized, when  $\tan R = 0$  and when  $\tan R = -\tan 2\alpha$ , the corresponding values of  $\tan 2(\theta - \alpha)$  being 0 and  $-\tan 4\alpha$ .  $\sin 2\gamma$  attains its maximum and minimum values  $\sin 2\alpha \cot 2\beta \tan r_m$ , when

$$\tan R = \tan r_m = \frac{1 \pm \sqrt{1 + \tan^2 2\alpha / \sin^2 2\beta}}{\tan 2\alpha / \sin^2 2\beta};$$

writing  $\tan 2\alpha / \sin 2\beta = \tan \psi$ ,  $\psi$  being a positive angle less than  $\pi$ ,

$$\tan r_m = \sin 2\beta \cot \psi/2, \text{ or } -\sin 2\beta \tan \psi/2,$$

and since  $\tan R = \sin 2\beta \tan \delta/2$ , the maximum and minimum values of  $\sin 2\gamma$  occur when  $\delta = n\pi - \psi$ .

The values of  $\tan R$  corresponding to the maximum and minimum values of  $\sin 2\gamma$  may be written

$$\tan R = \sin 2\beta \cdot x, \text{ where } x = \cot \psi/2 \text{ or } -\tan \psi/2,$$

$$\begin{aligned} \text{and } \tan 2\alpha &= -2 \sin 2\beta \cot \psi/2 / (1 - \cot^2 \psi/2) \\ &= 2 \sin 2\beta \tan \psi/2 / (1 - \tan^2 \psi/2) \\ &= -2 \sin 2\beta x / (1 - x^2); \end{aligned}$$

substituting these values in the expression for  $\tan 2(\theta - \alpha)$ , we find

$$\tan 2(\theta - \alpha) = 2 \sin 2\beta \frac{x(1+x^2)(1 - \cos 4\beta \cdot x^2)}{(1-x^2)(1+x^2)(1 - \cos 4\beta \cdot x^2)};$$

and unless  $x^2 = \sec 4\beta$ ,

$$\tan 2(\theta - \alpha) = 2 \sin 2\beta \frac{x}{1-x^2} = -\tan 2\alpha.$$

But when  $x^2 = \sec 4\beta$ ,

$$\tan 2\alpha = \sqrt{\cot^2 2\beta - 1}, \text{ or } \tan^2 2\beta = \cos^2 2\alpha;$$

hence corresponding to the maximum and minimum values of  $\sin 2\gamma$ , we have  $\tan 2(\theta - \alpha) = -\tan 2\alpha$ , unless  $\cos 2\alpha = \pm \tan 2\beta$ , a case that will be considered later.

Further,  $\sin 2\gamma = \pm 1$ , only if

$$\tan \psi/2 = \sin 2\alpha \cos 2\beta, \text{ or } \cot \psi/2 = \sin 2\alpha \cos 2\beta,$$

that is if  $\cos 2\alpha = \pm \tan 2\beta$ , and consequently, except in these cases,  $\theta$  determines the plane of maximum polarization.

Hence, reserving for future consideration the cases in which  $\cos 2\alpha = \pm \tan 2\beta$ , we see that as  $R$  increases from 0 to  $\pi$ , corresponding to a change of  $\delta$  from  $2n\pi$  to  $2(n \pm 1)\pi$  according as  $\beta$  is positive or negative, the light is initially plane polarized in the primitive plane of polarization; it then becomes elliptically polarized of a sign, the same as or opposite to that of the plate according as  $\sin 2\alpha$  is positive or negative: the ratio of the axes of the elliptic vibration attains its maximum value, the plane of maximum polarization being then either parallel or perpendicular to the principal section of the quartz, when  $\tan R = \sin 2\beta \cot \psi/2$  or  $= -\sin 2\beta \tan \psi/2$  according as  $\beta$  is positive or negative; when  $\tan R = -\tan 2\alpha$  the light is again plane polarized in



an azimuth symmetrical to the primitive plane of polarization with respect to the principal section of the plate; the sense of rotation then changes, and the ratio of the axes is again a maximum when  $\tan R = -\sin 2\beta \tan \psi/2$  or  $= \sin 2\beta \cot \psi/2$ , the plane of maximum polarization being again either parallel or perpendicular to the principal section; and finally, when  $R = \pi$ , the light is plane polarized in the original plane\*.

4. Turning now to  $\tan 2(\theta - \alpha)$ , we see that this attains its maximum and minimum values  $\tan R_m$ , when

$$\tan R = \tan R_m \\ = \frac{-\sin 2\alpha \cos 2\alpha \pm \sqrt{(\cos^2 2\alpha - \tan^2 2\beta)(\sin^2 2\alpha + \tan^2 2\beta)}}{\cos^2 2\alpha - \sin^2 2\alpha - \tan^2 2\beta}.$$

This, however, only gives a real value for  $\tan R$ , if  $\cos^2 2\alpha > \tan^2 2\beta$ , and when this is the case, writing

$$\cos^2 2\alpha - \tan^2 2\beta = \cos^2 \chi,$$

where  $\chi$  is a positive angle less than  $\pi/2$ , we have

$$\tan R_m = \tan(\chi - 2\alpha) \text{ or } -\tan(\chi + 2\alpha),$$

the value of  $\sin 2\gamma$  in these two cases being  $\tan 2\beta / \cos 2\alpha$ .

Thus if  $\cos^2 2\alpha < \tan^2 2\beta$ , the plane of maximum polarization rotates continuously as  $R$  increases; while if  $\cos^2 2\alpha > \tan^2 2\beta$ , it oscillates between two extreme positions, making an angle  $\chi/2$  on either side of the principal section of the plate or the perpendicular plane according as  $\cos 2\alpha$  is positive or negative†.

The following tables give some of the principal cases of the changes of the plane of maximum polarization and of  $\sin 2\gamma$  as  $R$  increases from 0 to  $\pi$ ,  $\beta$  being taken as positive.

\* The changes in the value of  $\sin 2\gamma$  may also be conveniently traced from the formula

$$\sin 2\gamma = 2 \sin 2\alpha \cos 2\beta \sin \delta/2 \sin(\psi + \delta/2) / \sin \psi.$$

† This oscillation of the plane of maximum polarization was deduced by Croullebois (*Ann. de Ch. et de Phys.* [4] xxviii. p. 382 (1873)) for the special case of light initially polarized in the principal section of the quartz. He appears to have overlooked the fact that it requires in this case that  $\tan 2\beta$  should be less than unity or  $\beta < \pi/8$ , a condition found by Monnory (*J. de Phys.* [2] ix. p. 277 (1890)). The condition for the general case does not appear to have been previously given.

TABLE I.— $\cos^2 2\alpha > \tan^2 2\beta$ .

$$0 < \alpha < \pi/4.$$

	$\tan R.$	$\tan 2(\theta - \alpha).$	$\theta.$	$\sin 2\gamma.$
1.	0	0	$\alpha$	0
2.	$\tan(\chi - \alpha)$	$\tan(\chi - 2\alpha)$	$\frac{\chi}{2}$	$\frac{\tan 2\beta}{\cos 2\alpha}$
3.	$\frac{\tan^2 2\beta}{\sin 2\alpha \cos 2\alpha}$	0	$\alpha$	$\sin 2\left(\tan^{-1} \frac{\tan 2\beta}{\cos 2\alpha}\right)$
4.	$\sin 2\beta \cot \frac{\psi}{2}$	$-\tan 2\alpha$	0	$\sin 2\alpha \cos 2\beta \cot \frac{\psi}{2}$
5.	$\cot 2\alpha$	$-\tan 4\alpha$	$-\alpha$	$\sin 2\left(\tan^{-1} \frac{\tan 2\beta}{\cos 2\alpha}\right)$
6.	$-\tan(\chi + 2\alpha)$	$-\tan(\chi + 2\alpha)$	$-\frac{\chi}{2}$	$\frac{\tan 2\beta}{\cos 2\alpha}$
7.	$-\tan 2\alpha$	$-\tan 4\alpha$	$-\alpha$	0
8.	$-\sin 2\beta \tan \frac{\psi}{2}$	$-\tan 2\alpha$	0	$-\sin 2\alpha \cos 2\beta \tan \frac{\psi}{2}$
9.	0	0	$\alpha$	0

$$\pi/4 < \alpha < \pi/2.$$

	$\tan R.$	$\tan 2(\theta - \alpha).$	$\theta.$	$\sin 2\gamma.$
1.	0	0	$\alpha$	0
2.	$\sin 2\beta \cot \frac{\psi}{2}$	$-\tan 2\alpha$	$\frac{\pi}{2}$	$\sin 2\alpha \cos 2\beta \cot \frac{\psi}{2}$
3.	$-\tan 2\alpha$	$-\tan 4\alpha$	$\pi - \alpha$	0
4.	$\tan(\chi - 2\alpha)$	$\tan(\chi - 2\alpha)$	$\frac{\pi}{2} + \frac{\chi}{2}$	$\frac{\tan 2\beta}{\cos 2\alpha}$
5.	$\cot 2\alpha$	$-\tan 4\alpha$	$\pi - \alpha$	$\sin 2\left(\tan^{-1} \frac{\tan 2\beta}{\cos 2\alpha}\right)$
6.	$-\sin 2\beta \tan \frac{\psi}{2}$	$-\tan 2\alpha$	$\frac{\pi}{2}$	$-\sin 2\alpha \cos 2\beta \tan \frac{\psi}{2}$
7.	$\frac{\tan^2 2\beta}{\sin 2\alpha \cos 2\alpha}$	0	$\alpha$	$\sin 2\left(\tan^{-1} \frac{\tan 2\beta}{\cos 2\alpha}\right)$
8.	$-\tan(\chi + 2\alpha)$	$-\tan(\chi + 2\alpha)$	$\frac{\pi}{2} - \frac{\chi}{2}$	$\frac{\tan 2\beta}{\cos 2\alpha}$
9.	0	0	$\alpha$	0

TABLE II.— $\cos^2 2\alpha < \tan^2 2\beta$ .

$$0 < \alpha < \pi/4.$$

	$\tan R.$	$\tan 2(\theta - \alpha).$	$\theta.$	$\sin 2\gamma.$
1.	0	0	$\alpha$	0
2.	$\cot 2\alpha$	$-\tan 4\alpha$	$\frac{\pi}{2} - \alpha$	$\sin 2\left(\tan^{-1} \frac{\tan 2\beta}{\cos 2\alpha}\right)$
3.	$\sin 2\beta \cot \frac{\psi}{2}$	$-\tan 2\alpha$	$\frac{\pi}{2}$	$\sin 2\alpha \cos 2\beta \cot \frac{\psi}{2}$
4.	$\frac{\tan^2 2\beta}{\sin 2\alpha \cos 2\alpha}$	0	$\frac{\pi}{2} + \alpha$	$\sin 2\left(\tan^{-1} \frac{\tan 2\beta}{\cos 2\alpha}\right)$
5.	$-\tan 2\alpha$	$-\tan 4\alpha$	$\pi - \alpha$	0
6.	$-\sin 2\beta \tan \frac{\psi}{2}$	$-\tan 2\alpha$	$\pi$	$-\sin 2\alpha \cos 2\beta \tan \frac{\psi}{2}$
7.	0	0	$\pi + \alpha$	0

$$\pi/4 < \alpha < \pi/2.$$

	$\tan R.$	$\tan 2(\theta - \alpha).$	$\theta.$	$\sin 2\gamma.$
1.	0	0	$\alpha$	0
2.	$\sin 2\beta \cot \frac{\psi}{2}$	$-\tan 2\alpha$	$\frac{\pi}{2}$	$\sin 2\alpha \cos 2\beta \cot \frac{\psi}{2}$
3.	$-\tan 2\alpha$	$-\tan 4\alpha$	$\pi - \alpha$	0
4.	$\frac{\tan^2 2\beta}{\sin 2\alpha \cos 2\alpha}$	0	$\frac{\pi}{2} + \alpha$	$\sin 2\left(\tan^{-1} \frac{\tan 2\beta}{\cos 2\alpha}\right)$
5.	$-\sin 2\beta \tan \frac{\psi}{2}$	$-\tan 2\alpha$	$\pi$	$-\sin 2\alpha \cos 2\beta \tan \frac{\psi}{2}$
6.	$\cot 2\alpha$	$-\tan 4\alpha$	$\frac{3\pi}{2} - \alpha$	$\sin 2\left(\tan^{-1} \frac{\tan 2\beta}{\cos 2\alpha}\right)$
7.	0	0	$\pi + \alpha$	0

5. The transition from one of these cases to the other occurs when  $\cos^2 2\alpha = \tan^2 2\beta$ , and then

$$\tan 2(\theta - \alpha) = \frac{2 \tan R(1 - \tan 2\alpha \tan R)}{1 - \tan^2 2\alpha \tan^2 R} = \frac{2 \tan R}{1 + \tan 2\alpha \tan R}.$$

In this case  $\theta$  increases from  $\alpha$  to  $\alpha + \pi/2$ , as  $R$  increases from 0 to  $\pi$ : the light is circularly polarized when  $\tan R = \cot 2\alpha$ , and while for values of  $R$  less than that given by this equation  $\theta$  gives the plane of maximum polarization, for greater values it determines the plane of minimum polarization\*.

\* When  $\tan R = \cot 2\alpha$ ,  $\tan 2(\theta - \alpha)$  of course becomes indeterminate, but it is easy to see that the change from determining the plane of maximum polarization to giving that of minimum polarization, or *vice versa*, occurs as  $\theta$  passes through the values  $(2n+1)\pi/4$ .

The ratio of the axes of the elliptic vibrations may be easily determined from the formula

$$\tan^2 \gamma = \tan(\alpha + \theta) \tan(\alpha + R - \theta)$$

which in this case reduces to the simple form

$$\tan \gamma = \pm \tan(\theta + \alpha) \tan(\theta - \alpha),$$

the sign to be attributed to  $\tan \gamma$  being determined from that of  $\sin 2\gamma$ .

The following table gives the principal changes relating to this case.

TABLE III.— $\cos^2 2\alpha = \tan^2 2\beta$ .

$$0 < \alpha < \pi/4.$$

	$\tan R.$	$\tan 2(\theta - \alpha).$	$\theta.$	$\tan \gamma.$
1.	0	0	$\alpha$	0
2.	$\cot 2\alpha$	$\cot 2\alpha$	$\frac{\pi}{4}$	1
3.	$-\cot 2\alpha$	$\infty$	$\frac{\pi}{4} + \alpha$	$\tan\left(\frac{\pi}{4} + 2\alpha\right)$
4.	$-\tan 2\alpha$	$-\tan 4\alpha$	$\frac{\pi}{2} - \alpha$	$\infty$
5.	$-\frac{\sin 2\alpha \cos 2\alpha}{1 + \cos^2 2\alpha}$	$-\tan 2\alpha$	$\frac{\pi}{2}$	$-\cot^2 \alpha$
6.	0	0	$\frac{\pi}{2} + \alpha$	$\infty$

$$\pi/4 < \alpha < \pi/2.$$

	$\tan R.$	$\tan 2(\theta - \alpha).$	$\theta.$	$\tan \gamma.$
1.	0	0	$\alpha$	4
2.	$-\frac{\sin 2\alpha \cos 2\alpha}{1 + \cos^2 2\alpha}$	$-\tan 2\alpha$	$\frac{\pi}{2}$	$\cot^2 \alpha$
3.	$-\tan 2\alpha$	$-\tan 4\alpha$	$\pi - \alpha$	0
4.	$-\cot 2\alpha$	$\infty$	$\frac{\pi}{4} + \alpha$	$-\tan\left(\frac{\pi}{4} + 2\alpha\right)$
5.	$\cot 2\alpha$	$\cot 2\alpha$	$\frac{3\pi}{4}$	-1
6.	0	0	$\frac{\pi}{2} + \alpha$	$-\infty$

If  $\alpha$  be between  $\pi/8$  and  $3\pi/8$ , cases (3) and (4) in the above table are interchanged.

6. When the light is analysed in a plane making an angle  $\phi$  with the principal section of the plate, the intensity of the radiation characterized by  $(\theta, \gamma)$  is

$$I = c^2 \{ \cos^2 \gamma \cos^2 (\phi - \theta) + \sin^2 \gamma \sin^2 (\phi - \theta) \} \\ = \frac{1}{2} c^2 \{ 1 + \cos 2\gamma \cos 2(\phi - \theta) \}.$$

Now from (2) and (3)

$$\begin{aligned} \cos 2\gamma \sin 2\theta \cos R - \cos 2\gamma \cos 2\theta \sin R \\ = \sin 2\gamma \cot \Delta = \sin (2\alpha + R) \cos \Delta \\ \cos 2\gamma \sin 2\theta \sin R + \cos 2\gamma \cos 2\theta \cos R \\ = 1 - \tan \left( \alpha + \frac{R}{2} \right) \sin 2\gamma / \sin \Delta \\ = 1 - \tan \left( \alpha + \frac{R}{2} \right) \sin (2\alpha + R) = \cos (2\alpha + R); \end{aligned}$$

whence

$$\begin{aligned} \cos 2\gamma \sin 2\theta &= \cos 2\alpha \sin R \cos R (1 + \cos \Delta) \\ &\quad + \sin 2\alpha (\cos^2 R \cos \Delta - \sin^2 R) \\ \cos 2\gamma \cos 2\theta &= \cos 2\alpha (\cos^2 R - \sin^2 R \cos \Delta) \\ &\quad - \sin 2\alpha \sin R \cos R (1 + \cos \Delta). \end{aligned}$$

Now

$$\begin{aligned} 1 + \cos \Delta &= \frac{2}{1 + \tan^2 \Delta / 2} = \frac{2}{1 + \cot^2 2\beta \sin^2 R} \\ &= \frac{2}{\cos^2 R + \sin^2 R / \sin^2 2\beta}, \end{aligned}$$

$\therefore$

$$\begin{aligned} \sin R \cos R (1 + \cos \Delta) &= \frac{2 \tan R}{1 + \tan^2 R / \sin^2 2\beta} = \sin 2\beta \sin \delta \\ \cos^2 R \cos \Delta - \sin^2 R &= \frac{2}{1 + \tan^2 R / \sin^2 2\beta} - 1 \\ &= 2 \cos^2 \frac{\delta}{2} - 1 = \cos \delta \\ \cos^2 R - \sin^2 R \cos \Delta &= 1 - \frac{2 \tan^2 R}{1 + \tan^2 R / \sin^2 2\beta} \\ &= 1 - 2 \sin^2 2\beta \sin^2 \frac{\delta}{2} \\ &= \cos^2 2\beta + \sin^2 2\beta \cos \delta. \end{aligned}$$

Hence

$$\begin{aligned}\cos 2\gamma \cos 2\theta &= \cos 2\alpha (\cos^2 2\beta + \sin^2 2\beta \cos \delta) - \sin 2\alpha \sin 2\beta \sin \delta \\ &= \cos 2\alpha \cos^2 2\beta + \sin 2\beta \sqrt{\sin^2 2\alpha + \cos^2 2\alpha \sin^2 2\beta} \\ &\quad \cos(\delta + \psi) \\ \cos 2\gamma \sin 2\theta &= \cos 2\alpha \sin 2\beta \sin \delta + \sin 2\alpha \cos \delta \\ &= \sqrt{\sin^2 2\alpha + \cos^2 2\alpha \sin^2 2\beta} \sin(\delta + \psi),\end{aligned}$$

where  $\tan \psi = \tan 2\alpha / \sin 2\beta$ ,

and

$$\begin{aligned}I &= \frac{1}{2}c^2 [1 + \cos 2\alpha \cos^2 2\beta \cos 2\phi \\ &\quad + \sqrt{\sin^2 2\alpha + \cos^2 2\alpha \sin^2 2\beta} \{ \sin 2\beta \cos 2\phi \cos(\delta + \psi) \\ &\quad + \sin 2\phi \sin(\delta + \psi) \}] \\ &= \frac{1}{2}c^2 [1 + \cos 2\alpha \cos^2 2\beta \cos 2\phi \\ &\quad + \sqrt{\sin^2 2\alpha + \cos^2 2\alpha \sin^2 2\beta} \sqrt{\sin^2 2\phi + \cos^2 2\phi \sin^2 2\beta} \\ &\quad \cos(\delta + \psi - \psi')],\end{aligned}$$

where  $\tan \psi' = \tan 2\phi / \sin 2\beta$ .

This formula contains the theory of the interference phenomena that are obtained when a conical pencil of polarized light traverses the plate and is subsequently analysed.

It may also be employed for the case in which the stream that passes through the polarizer, quartz plate and analyser is a parallel pencil of white light that is investigated with a spectroscope. With a sufficiently thick plate,  $\beta$  may be regarded as constant over a region of the spectrum corresponding to a variation of  $\delta$  by several multiples of  $\pi$ , and we see that the minima of intensity occur for the radiations for which  $\delta = (2n + 1)\pi + \Psi$ , where

$$\tan \Psi = \frac{\sin 2\beta (\tan 2\phi - \tan 2\alpha)}{\sin^2 2\beta + \tan 2\phi \tan 2\alpha},$$

the intensity then being

$$\begin{aligned}I &= \frac{1}{2}c^2 \{ 1 + \cos 2\alpha \cos^2 2\beta \cos 2\phi \\ &\quad - \sqrt{1 - \cos^2 2\alpha \cos^2 2\beta} \sqrt{1 - \cos^2 2\beta \cos^2 2\phi} \}.\end{aligned}$$

Thus as the analyser is turned, the dark bands travel along the spectrum, changing their intensity as they progress: they are perfectly black when  $\phi = \pi/2 \pm \alpha$ , and are least marked when  $\phi = 0$  and  $\pi/2$ .



With a plate perpendicular to the optic axis,  $\beta = \pi/4$ , and

$$I = c^2 \cos^2 \frac{\delta - \Psi}{2} = c^2 \cos^2 \frac{\delta - 2(\phi - \alpha)}{2};$$

hence as the analyser is turned, the bands traverse the spectrum, retaining the same intensity.

When the plate is parallel to the optic axis,  $\beta = 0$  or is at any rate very small, and

$$I = \frac{1}{2} c^2 \{1 + \cos 2\alpha \cos 2\phi + \sin 2\alpha \sin 2\phi \cos \delta\};$$

when  $\phi = 0$  or  $\pi/2$ , there are no bands, and in other cases the bands occur where  $\delta = (2n+1)\pi$  or  $2n\pi$ , according to the sign of  $\sin 2\alpha \sin 2\phi$ . Thus when  $0 < \alpha < \pi/2$ , the bands occur at the points for which  $\delta = (2n+1)\pi$  so long as  $0 < \phi < \pi/2$ , their intensity being zero, when  $\phi = \pi/2 - \alpha$ ; and as  $\phi$  increases through  $\pi/2$ , they change their position to that of the former maxima of intensity, becoming perfectly black when  $\phi = \pi/2 + \alpha^*$ .

7. Let us now suppose that the light emergent from the plate is examined with a Savart's analyser.

On leaving the quartz, the stream may be represented by the components

$$(u, v) = c(\cos \gamma, -\iota \sin \gamma) \text{Exp} \{\iota(nt + \epsilon)\}$$

polarized in planes making angles  $\theta$  and  $\theta + \pi/2$  with the principal section of the plate. Hence if  $\mu$  be the angle that the principal section of the quartz makes with that of the first plate of the analyser, the stream emerging from the Savart's plate will have components

$$\xi = \{\cos \gamma \cos(\theta + \mu) + \iota \sin \gamma \sin(\theta + \mu)\} \text{Exp} \{\iota(nt + \epsilon' + D/2)\}$$

$$\eta = \{\cos \gamma \sin(\theta + \mu) - \iota \sin \gamma \cos(\theta + \mu)\} \text{Exp} \{\iota(nt + \epsilon' - D/2)\}$$

polarized in the principal sections of the first and second plate, where  $D = D_2 - D_1$ ,  $D_1, D_2$  being the relative retardations of phase introduced by the two plates.

\* The above results agree with Beaulard's description of the phenomena (*J. de Phys.* (3) ii. p. 399 (1893)). Croullebois, however, gives a different account of the phenomena in the first and last cases (*loc. cit.* p. 391).

On analysis in an azimuth  $\phi$ , these give a stream

$$\begin{aligned} E &= \xi \cos \phi + \eta \sin \phi \\ &= [\{\cos \gamma \cos (\theta + \mu) + \iota \sin \gamma \sin (\theta + \mu)\} \cos \phi \text{Exp} (\iota D/2) \\ &\quad + \{\cos \gamma \sin (\theta + \mu) - \iota \sin \gamma \cos (\theta + \mu)\} \sin \phi \text{Exp} (-\iota D/2)] \\ &\quad \times \text{Exp} \{\iota (nt + \epsilon')\}, \end{aligned}$$

and the intensity, obtained by multiplying this by the conjugate expression, is

$$\begin{aligned} I &= \{\cos^2 \gamma \cos^2 (\theta + \mu) + \sin^2 \gamma \sin^2 (\theta + \mu)\} \cos^2 \phi \\ &\quad + \{\cos^2 \gamma \sin^2 (\theta + \mu) + \sin^2 \gamma \cos^2 (\theta + \mu)\} \sin^2 \phi \\ &\quad + \{\cos 2\gamma \sin 2(\theta + \mu) \cos D - \sin 2\gamma \sin D\} \sin \phi \cos \phi \\ &= \frac{1}{2} [1 + \cos 2\gamma \cos 2(\theta + \mu) \cos 2\phi \\ &\quad + \sin 2\phi \{\cos 2\gamma \sin 2(\theta + \mu) \cos D - \sin 2\gamma \sin D\}] \end{aligned}$$

and when, as is ordinarily the case,  $\phi = \pi/4$

$$I = \frac{1}{2} \{1 + \cos 2\gamma \sin 2(\theta + \mu) \cos D - \sin 2\gamma \sin D\}.$$

In order, then, that the bands may disappear, we must have

$$\sin 2\gamma = 0 \quad \text{and} \quad \sin 2(\theta + \mu) = 0;$$

that is, the light emergent from the quartz must be plane polarized in the principal section of one of the plates of the Savart's analyser—a result that is obvious from elementary considerations.

Now the light on leaving the quartz is plane polarized (1) when  $\delta = 2n\pi$ , the plane of polarization being then the same as it was initially; (2) when  $\tan \delta/2 = -\tan 2(\alpha - \mu)/\sin 2\beta$ ,  $\alpha$  being now the azimuth of the initial plane of polarization with respect to the principal section of the first plate of the analyser, and in this case  $\tan 2(\theta + \mu - \alpha) = -\tan 4(\alpha - \mu)$ . Hence in the first case, in order that the bands may disappear, the light must be initially polarized in one of the principal sections of the Savart's plate; and in the second case the principal section of the quartz must be in the azimuth  $\mu = k\pi/4 + \alpha/2$ .

It follows, then, that if the initial polarization be such that there are no bands before the introduction of the quartz plate, there will be a disappearance of the bands whenever the

relative retardation of phase due to the quartz plate is  $\delta = 2n\pi$ , and if the principal section of the quartz bisect the angle between the principal sections of the Savart's plate \*, there will be a further disappearance whenever  $\delta = (2n+1)\pi$ .

These results afford a method of setting the principal section of a quartz plate perpendicular to the axis of a spectrometer and of determining the error in the cutting of a plate that is supposed to be normal to the optic axis.

The collimator is furnished with a web, placed by the ordinary method perpendicular to the axis of the instrument. The telescope is turned to one side, and a telescope with a Savart's analyser is inserted in its place with its axis in the direction of that of the collimator, and the analyser is adjusted so that, when plane polarized light passes, the fringes are parallel to the web of the collimator. This sets the plane bisecting the angle between the principal sections of the plates of the analyser perpendicular to the turning axis of the spectrometer. This adjustment being made, the polarizer is turned until the fringes disappear exactly in the centre of the field.

The quartz plate is then placed at the centre of the table of the instrument, and by means of the telescope of the spectrometer its faces are made parallel to the turning axis, and the reading determined for which its normal is along the axis of the collimator.

With the polarizer adjusted as above, the fringes will in general be restored when the light traverses the quartz, but a series of positions can be found for which they vanish over a narrow streak extending across the field, and if the quartz plate be turned about its normal so as to bring its optic axis perpendicular to the turning axis of the spectrometer, there will be, intermediate to these, further positions at which the fringes are absent over a small area at the centre of the field.

In this way the principal section of the quartz is set perpendicular to the turning axis of the instrument, and a

\* The second of the above cases becomes identical with first when  $a=0$ , if  $k$  be even, and when  $a=\pi/2$ , if  $k$  be odd, since the value of  $\delta$  then becomes  $2n\pi$ .

determination of the angles of incidence, for which the centres of the zones of disappearance of the fringes are at the centre of the field, will give by calculation or by the graphic method employed by Beaulard \* the direction of the optic axis with respect to the faces of the quartz.

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XXXVI. *The Use of the Potentiometer on Alternate Current Circuits.* By CHARLES V. DRYSDALE, D.Sc.†

ONE of the greatest inconveniences in connexion with alternate current measurements has always been the question of range. Owing to the fact that all instruments employed for direct indication of alternate currents or P.D. have a square law, their range is small and they become impracticably delicate when low voltages are concerned. There is therefore a very great need for some instrument which, like the direct-current potentiometer, should be capable of measuring P.Ds. and currents of any range with accuracy. It would also be a great convenience if this instrument were capable of indicating the phase as well as the magnitude of the P.D. or current tested.

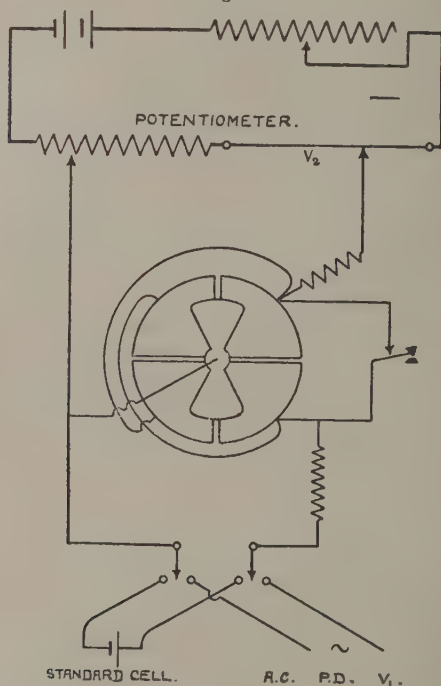
In attempting to apply the potentiometer principle to alternate current measurements, two processes seem to be possible—(a) the balancing of the alternate current P.D. against an equal direct current P.D. by the employment of some differential balancing device preferably of an electrostatic character, or (b) the balancing of two alternate current P.Ds. against one another. Such propositions as have hitherto been made for an alternate-current potentiometer have, the writer believes, always been based on the former idea, owing to the fact that the latter principle would involve equality of phase as well as of P.D. in the two voltages compared. He understands that some time ago Mr. Swinburne suggested the use of a differential electrometer in connexion with a potentiometer, but has not seen any

\* *Thèse de Doctorat*, Marseille, 1893.

† Read January 22, 1903.

details. An obvious device would be that shown in fig. 1, in which the P.D. to be measured is applied between the needle and one pair of quadrants, while that derived from the potentiometer is connected between the needle and the other pair of quadrants, in each case through a considerable

Fig. 1.



resistance, say one megohm. By connecting a key to the opposite pairs of quadrants, which normally short-circuits them, but disconnects them when depressed, a deflexion should be obtained to right or left proportional to  $V_1^2 - V_2^2$  or  $2VdV$  when balance is nearly obtained. Any want of symmetry in the electrometer should be eliminated by connecting the standard cell as shown, so as to obtain a double weighing method.

The procedure in this case is very simple, and exactly

resembles ordinary direct-current potentiometry. The sole difficulty lies in the electrometer. It should be possible by multicellular devices, or otherwise, to obtain a fair amount of sensitiveness with 1 or  $1\frac{1}{2}$  volts, the P.D. of the standard cell, as has been done by Mr. Addenbrooke, but when we come to a tenth of a volt or less, as is so often necessary in current measurement, electrostatic devices seem hopeless. All other magnetic or thermal devices are incorrect in principle as requiring an appreciable current to operate them, which militates against the accuracy of the potentiometer; but by employing a low-resistance potentiometer, and a differential thermal device, such as could be made from the Duddell thermo-galvanometer employing two junctions and two heaters, or two crossed thermo-junctions in series, a sufficiently sensitive and accurate potentiometer could perhaps be made.

In view of the difficulty of making such devices, however, the writer has turned his attention to the second method, *i. e.* of balancing the alternate current P.D. to be measured against a known alternate current P.D., by the interposition of a sensitive alternate-current detector, such as a telephone or vibration galvanometer. This of course implies that the two voltages under comparison must have the same magnitude, frequency, and phase, and, approximately, the same waveform\*.

The equality of frequency is of course secured by deriving the test and comparison voltages from the same source of supply, but some special device is needed for bringing the phases into coincidence. In some experiments recently made with the writer's phase-shifting transformer, however †, the accuracy of the phase-variation was found to be so good that it was thought worth while to introduce it into potentiometer work; and fig. 2 shows the diagram of the connexions which have been employed. The phase-shifter is here shown supplied with single phase current through a phase-splitting

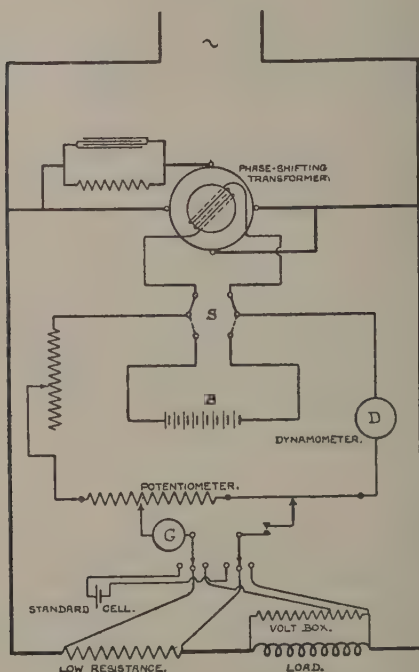
\* Since this was written the writer has been informed that Dr. Sumpner has somewhere suggested that an alternating-current potentiometer could be made, if any means were available for bringing the phase of the current in the slide-wire into coincidence with that of the P.D. to be measured.

† 'Electrician,' Dec. 11, 1908.



device, which can be built up with it so that it merely requires connecting straight on to the mains. The current derived from the secondary of this transformer traverses the rheostat coils, and slide-wire of an ordinary potentiometer, such as the Crompton form; and also some form of dynamometer-current measuring device. The writer has found

Fig. 2.



a Weston A. C. *voltmeter* excellent for this purpose, as it gives a very good reading with the current (.05 ampere) employed with the Crompton potentiometer. The remainder of the potentiometer is connected up precisely as for ordinary direct current working, except that a vibration galvanometer is substituted for the ordinary form. By having an auxiliary battery B and throw-over switch S, the arrangement is

always ready for either direct-current or alternate-current comparisons.

The procedure is then as follows :—The throw-over switch is first arranged to supply direct current from the battery B, and the rheostat altered until the standard cell is balanced at its nominal value, as in ordinary direct-current working. The dynamometer reading is then carefully noted, or a fiducial mark is made, which always represents the exact .05 ampere. On now throwing over to alternate current, the rheostat is altered to reproduce the same R.M.S. current as indicated by the dynamometer D ; and the position of the potentiometer contact is shifted and the phase-shifter turned, until the vibration galvanometer shows no vibration. The adjustment is very similar to that of the two resistances in an Anderson's bridge, and presents no difficulties. The voltage and phase are then read off on the potentiometer and phase-shifter respectively. In the illustration a low resistance is connected in series with the load, and a volt-box in parallel with it, and the current and P.D. can be determined directly in the ordinary way. Of course the phase-shifting transformer must always be connected to the same source of supply as the load.

Instead of the vibration galvanometer, a dynamometer or electrometer in which one pair of terminals is connected to the supply so as to be "separately excited," and the other to the ordinary galvanometer terminals, has been employed with good results. The deflexions are then to left or to right as in an ordinary galvanometer. But it must not be forgotten in this case that balance will be obtained not only when the vectors of the two P.Ds. compared are coincident, but also when their vector difference is in quadrature with the P.D. of the supply. For this purpose, if such an instrument is used it must have its "exciting terminals" changed over from one phase to the other, and balance secured in both cases.

Before giving examples of readings obtained in this manner, we must consider what is really measured. It is obvious that this device does not indicate the effective or R.M.S. value of the P.D., except when the supply and tested wave-forms are both sinusoidal or of identical form. When

a vibration galvanometer is employed, its sensitiveness to the fundamental wave is so great in comparison with that to the harmonics, that we shall be practically correct in assuming that it is the fundamental wave only which is measured, and the comparison is really between the mean and not the R.M.S. values. On the other hand, if there is any serious difference of wave-form, the sensitiveness to the upper harmonics should be sufficient to prevent an exact balance being secured, and this warns us when great accuracy is not to be expected.

The effective value of an irregular periodic voltage is

$$\bar{V} = \sqrt{\sum \bar{V}_n^2},$$

where  $\bar{V}_n$  is the effective value of the harmonic of order  $n$ . If the sum of the squares of the higher harmonics is small in comparison with the square of the fundamental, we may take the approximate square root, and

$$\bar{V} = \bar{V}_1 + \frac{1}{2\bar{V}_1} \sum \bar{V}_n^2,$$

or

$$\bar{V} = \bar{V}_1 \left\{ 1 + \frac{1}{2\bar{V}_1^2} \sum \bar{V}_n^2 \right\},$$

where  $\sum \bar{V}_n^2$  in the last two expressions implies the sum of the squares of the effective values of all the harmonics, excluding the fundamental.

In the case of a single harmonic this becomes

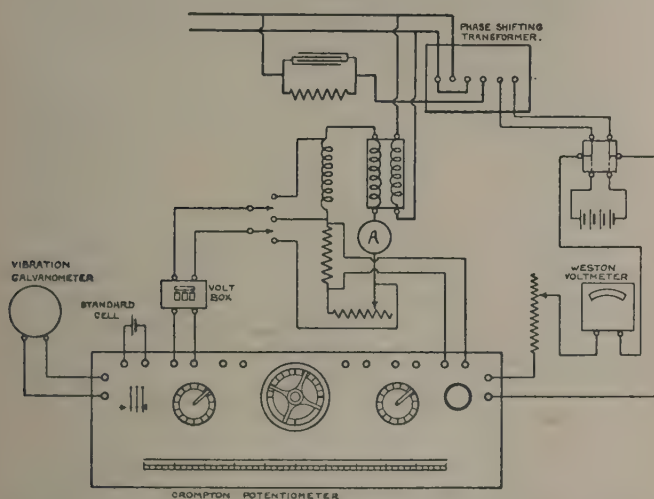
$$\bar{V} = \bar{V}_1 \left\{ 1 + \frac{1}{2} \left( \frac{\bar{V}_n}{\bar{V}_1} \right)^2 \right\},$$

and the expression in brackets only differs 1 per cent. from unity when  $\bar{V}_n$  is 14 per cent. of  $\bar{V}_1$ . Such a difference should be fairly easily recognizable, and would probably be sufficient to prevent the obtaining of a perfect balance on the vibration galvanometer; and, moreover, if the distortion of wave-shape is great, there is little object, as a rule, in great accuracy of measurement. If accuracy is required in such cases, it can of course be secured by tracing the wave-shape and making the above correction, but if care is taken to have a supply of fairly sinusoidal wave-form, the measurements

may be considered sufficiently accurate for the majority of purposes. It should be observed that this difficulty, or a similar one, is inseparable from any method which can be devised, in which the detector is employed to indicate the *difference* between the test and potentiometer P.D.s, as is done with direct currents.

*Measurements.*—Two illustrations will be sufficient to show the possibilities of the method. An ordinary Crompton potentiometer was used with a Weston dynamometer voltmeter in series with it, and the connexions were made as in fig. 3. A Rubens vibration galvanometer was at first used,

Fig. 3.



but was replaced by a Campbell bifilar instrument, which was found much more sensitive and convenient. A check with direct current and a Cadmium cell showed that the normal current was obtained when the dynamometer indicated 82 "volts," and this reading was reproduced in all the alternate-current measurements.

The first test was a comparison of the readings of a Kelvin Deka-ampere-balance, with the current as given by the P.D. across a Crompton  $\cdot 04 \omega$  manganin resistance.

The following results were obtained :—

Current by Kelvin.	P.D. by Potentiometer.	Current by Potentiometer.
25	1·002	25·05
20	·8018	20·045
15	·6022	15·055

Frequency of supply 72 ~.

The readings were obtained with great ease and definiteness, half a division on the slide-wire, or ·0005 volt, being easily detected, in spite of the fact that trouble was found from mechanical vibration, as the alternator was in the same room. There seems no reason to doubt that ·0002 volt could be easily detected, or that a drop of ·1 volt could be measured to an accuracy of ·2 per cent. or closer. The readings of the phase angle repeated themselves constantly to within ·1 degree.

To give some idea of the accuracy of phase measurements, a resistance and ironless choking-coil were connected in series with the ·04  $\omega$  resistance, and an alternating current of 15 amps. at 60 ~ passed through the combination. Readings were then taken of the P.D. across the ·04  $\omega$  resistance, across the choking-coil, total resistance, and the whole circuit, the last three being taken by the aid of a 10  $\times$  1 volt box. The following are the results :—

Circuit.	First Test.		Second Test.	
	P.D.	Phase-angle, degrees.	P.D.	Phase-angle.
·04 resistance ...	·5960	46·5	.....	.....
Total resistance .	3·117	47·1	3·108	47·3
Choking-coil .....	4·880	125·9	4·928	125
Mains .....	6·185	95·3	6·218	94·9

The difference between the first and second readings is due

to the fact that a carbon resistance was employed which became heated.

Fig. 4 is a graphical representation of the result in the last case, the horizontal line representing the resistance drop, and the other two lines the drop across the choking-coil and whole circuit respectively. Although the triangle does not quite close, the result may be taken as satisfactory in consideration of the fact that single phase supply was used in

Fig. 4.



this case, and the phase was split by condensers. It should be noted also that errors in the phase angle do not in any way affect the accuracy of the magnitudes of the P.Ds.

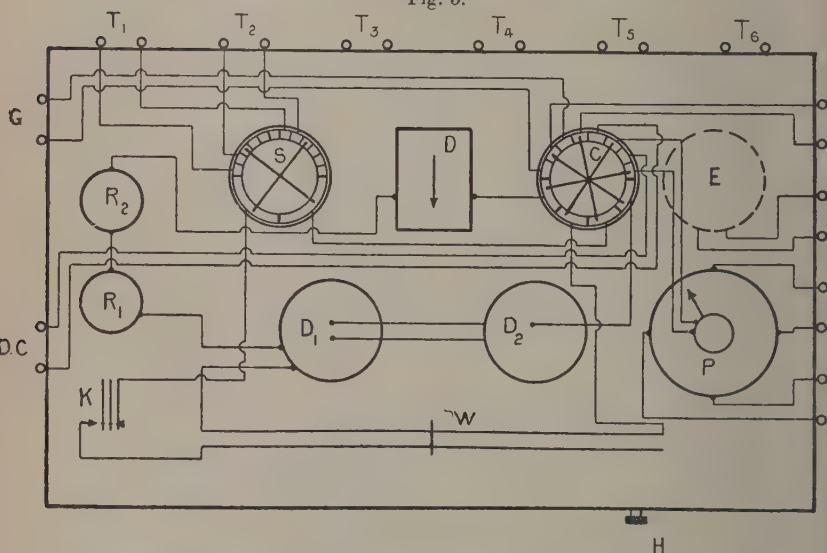
The writer has designed a potentiometer in which a small phase-shifting transformer and dynamometer are included in the case, with a throw-over switch, permitting the instrument to be used as either a direct or alternate current potentiometer.

Fig. 5 shows the connexions, which need little explanation. The main circuit of the potentiometer, consisting of the rheostats  $R_1$  and  $R_2$ , the two dials  $D_1$  and  $D_2$ , the side-wire  $W$ , and the dynamometer  $D$ , is connected to two of the blocks of the change-over switch  $C$ , which is similar in form to the well-known selector switch used in the Crompton Potentiometer. The derived P.D. from the dial  $D_2$  and slide-wire contact is led by the key  $K$  and the selector switch  $S$  to two more of the contacts on  $C$ . The remaining eight contacts on



C are connected in pairs to the phase-shifting transformer P, the battery terminals D.C., the ordinary galvanometer terminals G, and the vibration galvanometer terminals V.G. The phase-shifter primary is excited from the terminals A.C., which are here shown four in number, for two-phase or split-phase. A single motion of the switch C to right or left connects the main circuit of the potentiometer on to the battery or secondary of the transformer, and simultaneously

Fig. 5.



connects the sliding contacts through the selector switch to the appropriate galvanometer. The key K being provided with a holding-down cam, leaves the hands free to move the potentiometer contacts and rotate the phase-shifter simultaneously. A frequency meter F of the vibrating reed type may be fixed on the base if desired, in which case the instrument is available for inductance and capacity measurements.

The dynamometer D is not provided with a scale, but with a fiducial mark which can be slightly adjusted to suit the standard cell-check. It will also be made astatic to avoid risk of error from stray field.

This apparatus should be capable of giving very good

results. In the meantime the results already obtained appear to justify the belief that this method is capable of being used with considerable accuracy and convenience for all P.D., current, and phase measurements.

The numerous applications of such an instrument are obvious, and include P.D., current, phase, and power measurement over almost any range; inductance and capacity test, and determination of dielectric losses, &c. One of the most valuable applications, however, is to search-coil work with alternate current, whereby the distribution of main and leakage fluxes in cores, and the propagation of magnetic impulses can be studied, and the variation of current and P.D. along a cable fed with alternate currents.

The writer's thanks are heartily tendered to Mr. A. C. Jolley for setting up the apparatus and great assistance in the experiments.

#### DISCUSSION.

Mr A. CAMPBELL remarked that the first method suggested by the author had already been described by Mr. J. Swinburne (Phys. Soc. Dec. 1893). It was also used with success at the Reichsanstalt in an improved form by Dr. Drewall (*Zeitschr. für Instrumentenkunde*, April 1903). Dr. Drysdale's phase-turning method was interesting and would give good results in many cases. The vibration galvanometer, however, was so very much more sensitive for its own frequency than for others, that it would not be likely to give any indication of errors due to other harmonics unless they were very pronounced. Hence the method must be used with great caution.

Mr W. DUDELL expressed his interest in the paper, and asked if the limit of accuracy depended on the sensitiveness of the galvanometer and if this was limited by the back E.M.F. of the instrument. He pointed out that the harmonics could be investigated by tuning up the galvanometer to be in unison with them. The device by which the phase was changed without changing the current was very useful.

Mr RAYNER pointed out that the accuracy attainable depended on the sensitiveness of the Weston voltmeter employed. A vital point in the instrument was the phase-shifter with the sinusoidal windings. By using two galvanometers the fundamental and a harmonic could both be measured.

Mr. PATERSON joined with others in expressing appreciation of the paper. Dr. Drysdale claimed that low resistances dropping 0.1 volt could be used with this potentiometer. He presumed that the object of measuring the phase displacement between the two E.M.F.s was to obtain a value for the watts in the circuit. In this case the employment of low resistances was a great disadvantage on account of the large time

constant which these shunts necessarily have, even though constructed with the utmost care. He thought the failure of the vector diagram exactly to close was more likely to be due to some such error coming in to the measurement of the phase-angle than to any inaccuracy in the Kelvin Balance—as hinted by the Author.

The AUTHOR, in reply, said that he regretted that time had not permitted his making further investigations into previous work in this direction, but he had alluded to Mr Swinburne's suggestions, and had claimed no credit for the electrostatic device. Mr Campbell's criticisms mainly dealt with wave-form errors. Of course where iron cores at considerable saturation were employed, the distortion of wave-shape would be considerable, but with the single exception of power measurement which should then be made with a wattmeter, he knew of no object in such cases for accurate P.D. or current measurements. For ordinary standardisation of P.D. and current, this difficulty did not present itself, as there was no difficulty whatever in obtaining a sufficiently closely sinusoidal wave-form, and the wave-forms of the measured and balancing P.Ds. were usually similar, being derived from the same source. The use of a second vibration galvanometer tuned to the third harmonic would be excellent if it was desired to measure them, but he believed that it would rarely be necessary, and there should be no difficulty in making measurements to an accuracy of 0.1 per cent. with the ordinary alternators and resistances. For inductance and capacity measurements there was of course a positive advantage in an instrument which disregarded the harmonics. In reply to Mr Duddell, there was no doubt that the vibration galvanometer did give a back E.M.F., as was shown by a telephone connected to its terminals when the coil was set in motion, and this would probably increase the damping, but he did not know the amount of the influence on the sensitiveness. He was glad the importance of the sinusoidal winding of the phase-shifting transformer was appreciated, and hoped that it would be found effective in preserving perfect equality of the potentiometer current as the phase was varied. It was true that slots or tunnels were employed in the stator core, but they were of considerable number, and the rotor and stator pitches were incommensurable. As to accuracy, it was true that this was limited by the dynamometer instrument, but this was fairly sensitive as it was always used at its best range, and it was to be doubted whether it was possible under any ordinary circumstances to make alternate current measurements to within an accuracy of a few parts in ten thousand, owing to variations in the alternator speed or E.M.F. Finally, he was certain that no ordinary low resistance standards would give any appreciable inductive errors. To produce an error in magnitude of .1 per cent. would require a phase displacement of about 2.7 degrees, which was only slightly exceeded by helical iron wire resistances at 50~, and the straight strip resistances of the Reichsanstalt form were almost the perfection of non-inductive resistances. In any case inductance would not affect the closing of the vector triangle.

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XXXVIII. *Note on the Luminous Efficiency of a Black Body.* By CHARLES V. DRYSDALE, D.Sc.\*

At the present moment, when the question of light production is so much to the front, it is of interest to ascertain the possibilities of various substances as incandescent bodies. Until comparatively recently nearly all artificial illumination has been derived from the incandescence of carbon in some form or other, and consequently from a substance which approximates to a perfectly black body. It is therefore of interest to ascertain the possibilities of such a body as a light radiator at various temperatures, and this can be readily obtained from the radiation formulæ.

The best known and most convenient law for calculation is that of Wien †,

$$R_{\lambda} = C_1 \lambda^{-5} e^{-\frac{C_2}{\lambda T}}, \quad . . . . . (1)$$

where  $\lambda$  is the wave-length,  $T$  the absolute temperature,  $R_{\lambda}$  the intensity of radiation corresponding to the wave-length  $\lambda$ , and  $C_1$  and  $C_2$  constants. The value of  $C_2$  according to Paschen & Wanner ‡ should be 14,400, and to Lummer & Pringsheim § 14,700 when  $\lambda$  is in microns, and  $T$  in degrees centigrade.

Wien's law is not quite as exact as that of Max Planck ||

$$R_{\lambda} = C_1 \lambda^{-5} / (e^{\frac{C_2}{\lambda T}} - 1),$$

but only differs from it by 1 per cent. when  $\lambda T$  is greater than  $C_2/4.6$  or about 3000. It may therefore be used for most calculations.

The energy radiated between any two wave-lengths is therefore

$$\int_{\lambda_1}^{\lambda_2} R_{\lambda} d\lambda = \frac{C_1}{C_2} T \left[ \epsilon^{-\frac{C_2}{\lambda_1 T}} \left\{ \frac{1}{\lambda_1^3} + \frac{3T}{C_2 \lambda_1^2} + \frac{6T^2}{C_2^2 \lambda_1} + \frac{6T^3}{C_2^3} \right\} - \epsilon^{-\frac{C_2}{\lambda_2 T}} \left\{ \frac{1}{\lambda_2^3} + \frac{3T}{C_2 \lambda_2^2} + \frac{6T^2}{C_2^2 \lambda_2} + \frac{6T^3}{C_2^3} \right\} \right], \quad (2)$$

\* Read January 22, 1909.

† *Berl. Ber.* 9th February, 1893.

‡ *Berlin Akad. Sitzungsberichte*, ii. pp. 5-11, 1899.

§ *Verh. d. deutsch. Phys. Ges.* 1899, p. 23.

|| *Ann. der Phys.* i. 1900.

and the total radiation

$$R = \int_0^{\infty} R_{\lambda} d\lambda = 6 \frac{C_1}{C_2^4} T^4,$$

corresponding to the Stefan law

$$R = \sigma T^4. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence

$$6 \frac{C_1}{C_2^4} = \sigma \quad \text{and} \quad C_1 = \frac{\sigma C_2^4}{6}.$$

Several determinations of the constant  $\sigma$  have been made, but that of Kurlbaum \* is most generally accepted. His value is  $5.32 \times 10^{-12}$  watts per sq. cm. Combining this with Lummer & Pringsheim's value of 14700 for  $C_2$  we have  $C_1 = 41,400$ .

At the same time by logarithmic differentiation

$$\frac{1}{R_{\lambda}} \cdot \frac{\partial R_{\lambda}}{\partial \lambda} = -\frac{5}{\lambda} + \frac{C^2}{\lambda^2 T}$$

from which we have the well-known relation

$$\lambda_{\max.} T = \frac{C_2}{5} \quad \text{or} \quad \lambda_{\max.} = \frac{C_2}{5T},$$

where  $\lambda_{\max.}$  is the dominant wave-length.

Hence for numerical calculation we have the following formulæ:—

Total radiation

$$R = \int_0^{\infty} R_{\lambda} d\lambda = 5.32 \times 10^{-12} T^4 \text{ watts per sq. cm.} \quad (4)$$

$$R_{\lambda} = 41400 \lambda^{-5} e^{-\frac{14700}{\lambda T}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\lambda_{\max.} T = 2940. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\int_{\lambda_1}^{\lambda_2} R_{\lambda} d\lambda = 2.82T \left[ \lambda_2^{-4} e^{-\frac{14700}{\lambda_2 T}} \left( \frac{1}{\lambda_2^3} + 2.04 \times 10^{-4} \frac{T}{\lambda_2^2} + 2.78 \times 10^{-8} \frac{T^2}{\lambda_2} + 1.89 \times 10^{-12} T^3 \right) - \lambda_1^{-4} e^{-\frac{14700}{\lambda_1 T}} \left( \frac{1}{\lambda_1^3} + 2.04 \times 10^{-4} \frac{T}{\lambda_1^2} + 2.78 \times 10^{-8} \frac{T^2}{\lambda_1} + 1.89 \times 10^{-12} T^3 \right) \right]. \quad (7)$$

To obtain the radiant luminous efficiency we have only to

\* *Ann. Phys. Chem.* lxx. 4. pp. 746-748, 1898.

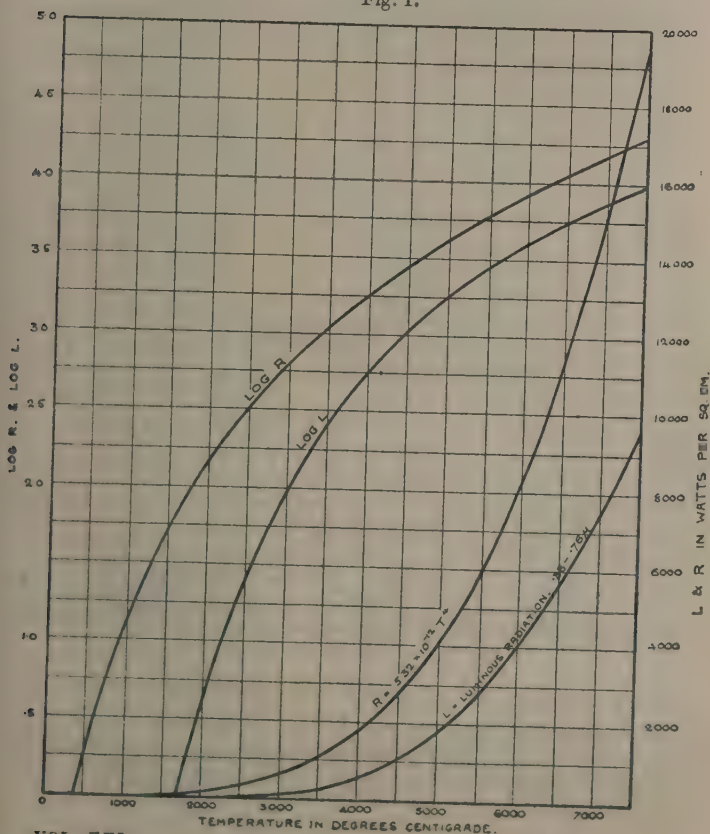
calculate the value of the last integral between the spectral limits and divide by the total radiation  $R$ , and we have

Radiant efficiency  $\eta_R$

$$= 5.3 \times 10^{11} \left[ \epsilon_{\lambda_2}^* - \frac{14700}{\lambda T} \left( \frac{1}{\lambda^2 T^3} + \frac{2.04 \times 10^{-4}}{\lambda^2 T^2} + \frac{2.78 \times 10^{-8}}{\lambda T} + 1.89 \times 10^{-12} \right) \right]. \quad (8)$$

The following results have been worked out by my Assistant Mr. A. F. Burgess, B.Sc., and are shown in the annexed table. In fig. 1 the values of the total and luminous ra-

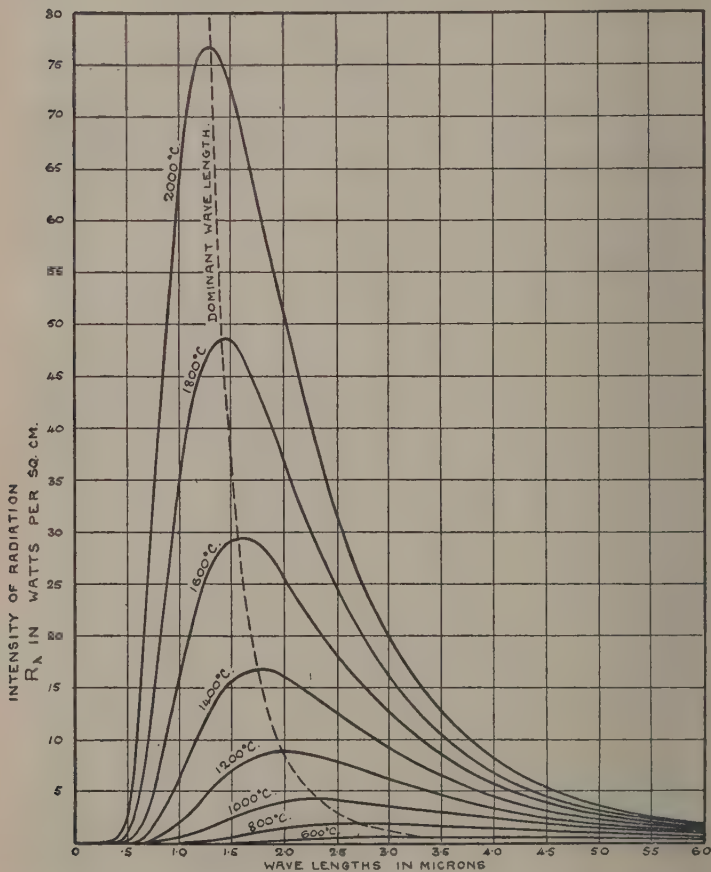
Fig. 1.





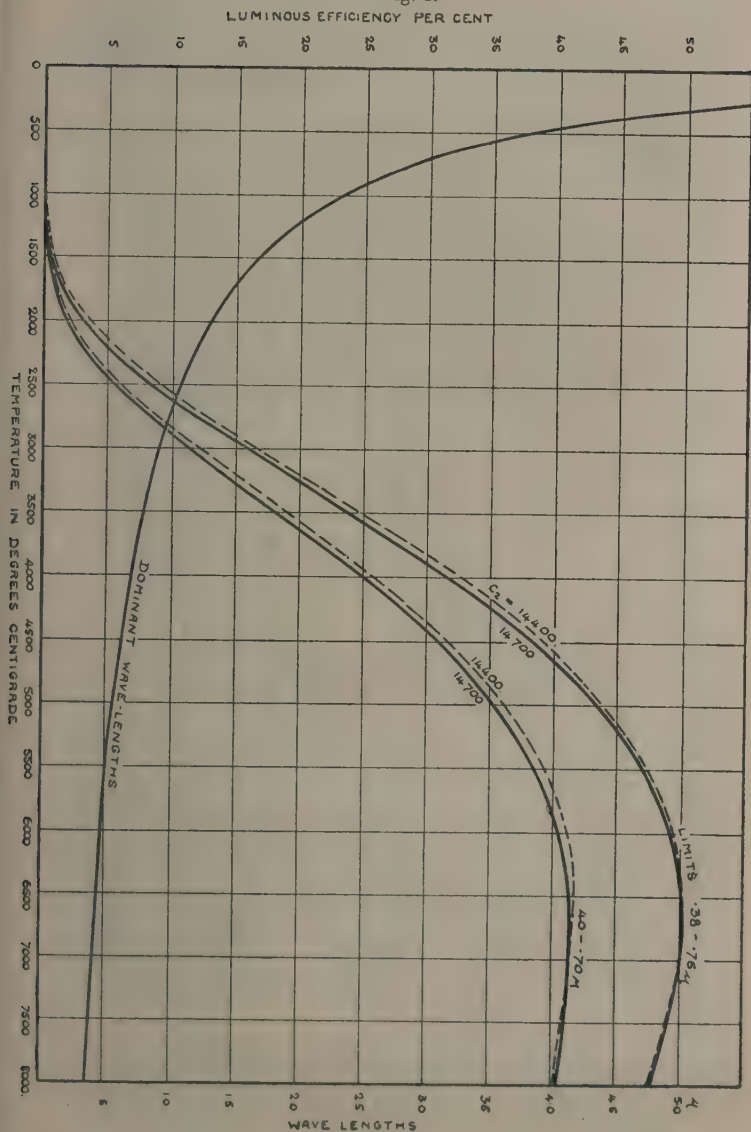
diation and their logarithms are plotted against the ordinary temperature. Fig. 2 shows the relation between the intensity of radiation in watts per sq. cm. and wave-length for various

Fig. 2.



temperatures on the centigrade scale from formula 5. Fig. 3 shows the relation between the dominant wave-length and temperature from (6) and the radiant efficiency  $\eta_R$  from (8) for two different values of the spectral limits, one being taken from '38

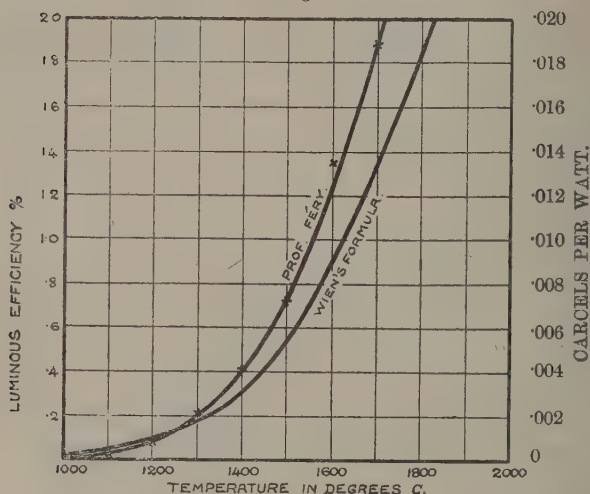
Fig. 3.



to  $\cdot 76\mu$ , the other from  $\cdot 4$  to  $\cdot 7\mu$ , the latter comprising practically the whole of the useful luminosity of the spectrum. The dotted curves on the same sheet correspond to Paschen & Wanner's value of  $C_2$  and do not differ much from the others.

Fig. 4 shows the comparison between the results so calculated and the experimental determinations of Prof. Féry\*,

Fig. 4.



who has found the amount of light radiated per watt of total radiation at various temperatures. Taking the Carcel as 10 c.p., and the proportion of light to total energy from the calculated curve, we find that the luminous energy per c.p. or the mechanical equivalent of light comes out at about  $\cdot 075$  watt per candle @  $1600^\circ$ , which is not far from the most probable result of other determinations. It should be mentioned that the employment of Planck's formula instead of Wien's would have given lower rather than higher values for the efficiency.

\* *Rayonnement Calorifique et Lumineux*. Thèses présentées à la Faculté des Sciences. Paris 1902.

These curves show at a glance the enormous extent to which the luminous efficiency is dependent upon temperature, and how extremely low it is at ordinary temperatures. At  $1500^{\circ}\text{C}$ . the efficiency is only of the order of 1 per cent. or less, while at  $2000^{\circ}\text{C}$ . it is still only about 3 per cent. In order to get anything like an efficient source of light by pure temperature radiation therefore, temperatures of from  $4000^{\circ}$  to  $6000^{\circ}\text{C}$ . would be required, and even then the efficiency would be only of the order of 30 to 50 per cent. This strongly points therefore to the necessity for working in the direction of selective radiation or luminescence.

It is perhaps worthy of note that the temperature of maximum efficiency is in the neighbourhood of the temperature of the sun as found by Prof. Féry, viz.  $5500^{\circ}\text{C}$ ., which is due to the close agreement of the dominant wave-length of the sun with that which produces the maximum luminous effect on the retina. This correspondence is doubtless to be attributed to the influence of natural selection on the eye, as Prof. Féry has himself pointed out.

Temp. $^{\circ}\text{C}$ .	$\lambda$ max. $\mu$ .	Total Radiation Watts/sq. cm.	Luminous Radiation Watts sq. cm.		Luminous Efficiency per cent.	
			$\cdot 4 - \cdot 7 \mu$ .	$\cdot 38 - \cdot 76 \mu$ .	$\cdot 4 - \cdot 7 \mu$ .	$\cdot 38 - \cdot 76 \mu$ .
1000 ...	2.31	13.97	$8.653 \times 10^{-4}$	$2.525 \times 10^{-3}$	0.062	0.181
1500 ...	1.66	52.57	1358	2766	2.58	526
2000 ...	1.29	142.01	2.534	4.264	1.784	3.003
2500 ...	1.06	314.5	17.47	25.96	5.554	8.253
3000 ...	.90	610.5	69.39	98.24	11.367	16.092
3500 ...	.78	1078.1	195.5	257.7	18.137	23.919
4000 ...	.69	1773.4	452.2	562.8	25.498	31.737
4500 ...	.615	2760.8	849.5	1060.3	30.771	38.405
5000 ...	.56	4112.5	1454.0	1789.0	35.355	43.499
5500 ...	.51	5909.0	2280.2	2779.6	38.589	47.041
6000 ...	.47	8237.3	3339.8	4051.0	40.544	49.180
6500 ...	.435	11194	4639.6	5606.3	41.446	50.081
7000 ...	.405	14883	6173.5	7442.5	41.477	50.003
7500 ...	.38	19428	7931.7	9536.2	40.827	49.087
8000 ...	.355	24921	10079	11920	40.449	47.835

### DISCUSSION.

The SECRETARY asked the Author if he would state exactly what he meant by the term "luminous efficiency." The ratio of luminosity to energy radiated was different in different parts of the spectrum.

Dr RUSSELL thought that the physiological effect on the eye ought to be considered. If the paper had been entitled "The radiation efficiency, for rays contained within the limits of the visible spectrum, of a black body" it would perhaps have been described better. The sensibility of the eye to light rays varied not only with the wave-length but also with the intensity of the rays and the time during which they had been acting on the retina. If we considered only the objective stimulus of the luminosity, the Author's implied definition might be, he thought, accepted. But if we consider, as we ought, that the subjective sensation produced in the normal eye has to be taken into account when judging luminous efficiency, another definition is required and the problem becomes far more complicated.

The AUTHOR, in reply, said that the term "luminous efficiency" was not very generally understood. There were three modes of defining it, of which two were due to Prof. Nichols of Cornell University, who distinguished between "total efficiency" and "radiant efficiency." The former of these quantities was the ratio of the luminous energy radiated between the spectral limits to the total energy consumption of the source of light, and the latter the ratio of the luminous energy as above to the total radiation from the source. As had just been pointed out, however, neither of these definitions took any cognizance of the very different luminosities of the various spectral colours, and this had been realized by Dr C. Guilleaume, who had proposed to correct the luminous radiation by multiplying each ordinate of the radiation curve by a factor depending on the luminosity at that ordinate, which was unity at the point of maximum effectiveness ( $54\mu$ ). The Author had proposed (Proc. R. S. 1908) the term "reduced" total or radiant efficiency, for the efficiency expressed on this basis, which was evidently the most rational one. The term "radiant luminous efficiency" was, however, recognized and had been employed correctly in the paper.

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PROCEEDINGS  
AT THE  
MEETINGS OF THE PHYSICAL SOCIETY  
OF LONDON.  
SESSION 1908-1909.

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February 28th, 1908.

Meeting held at the Imperial College of Science and Technology.

Dr. C. CHREE, F.R.S., President, in the Chair.

The following Papers were read :—

1. Contact Potential Differences determined by Null Solutions.  
By Messrs. S. W. J. SMITH and H. MOSS.
  2. An Experimental Examination of Gibbs's Theory of Surface Tension as the Basis of Adsorption, with an Application to the Theory of Dyeing. By Mr. W. C. M. LEWIS.
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March 13th, 1908.

Meeting held at the Imperial College of Science and Technology.

Dr. C. CHREE, F.R.S., President, in the Chair.

The following Candidates were elected Fellows of the Society :—

A. E. GARRETT, A. E. HALL, and W. R. WILLIAMS.

The following Papers were read :—

1. The Distribution in Electric Fields of the Active Deposits of Radium, Thorium, and Actinium. By Mr. S. RUSS.



2. A Note on certain Dynamical Analogues of Temperature Equilibrium. By Prof. G. H. BRYAN.
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March 27th, 1908.

Meeting held at the Northampton Institute, Clerkenwell, by the kind invitation of the Governing Body and the Principal.

Dr. C. CHREE, F.R.S., President, in the Chair.

The following Papers were read:—

1. Notes on the Plug Permeameter. By Dr. C. V. DRYSDALE.
2. The Use of Shunts and Transformers with alternate-current measuring Instruments. By Dr. C. V. DRYSDALE.
3. Wattmeters. By Dr. C. V. DRYSDALE.

An Experimental Demonstration of Alternate Current Wave Propagation in a Helix was given by Dr. C. V. DRYSDALE.

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April 10th, 1908.

Meeting held at the Imperial College of Science and Technology.

Dr. C. CHREE, F.R.S., President, in the Chair.

The following Candidates were elected Fellows of the Society:—  
S. D. CHALMERS and E. N. LADLER.

The following Papers were read:—

1. An Experimental Investigation of the nature of Gamma Rays. By Prof. W. H. BRAGG and Mr. MADSEN.
  2. Experiments on Artificial Fulgurites. By Miss D. D. BUTCHER.
  3. Short-Spark Phenomena. By Mr. W. DUDELL.
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May 8th, 1908.

Meeting held at the Imperial College of Science and Technology.

Dr. C. CHREE, F.R.S., President, in the Chair.

The following Papers were read:—

1. A Modified Theory of Gravitation. By Dr. C. V. BURTON.
2. An examination of the Formulæ for Grading of Cables. By Mr. C. S. WHITEHEAD.
3. Illustrations of Geometrical Optics. By Mr. R. M. ARCHER.

May 22nd, 1908.

Meetings held at the Imperial College of Science and Technology.

Dr. C. CHREE, F.R.S., President, in the Chair.

A Special General Meeting was held at which the following Resolution was passed:—

*Resolution*:—"That the Articles of Association contained in the printed document submitted to the Meeting, and, for the purpose of identification, subscribed by the Chairman thereof, be, and the same are hereby adopted as the Articles of Association of the Society."

An Ordinary Meeting of the Society was then held.

The following Papers were read:—

1. The Spectrum Top. By Mr. F. P. SEXTON.
2. The Coefficient of Diffusion. By Mr. B. W. CLACK.
3. The Production of Small Alternating Currents of Variable Frequency. By Mr. B. S. COHEN.

June 12th, 1908.

Meetings held at the Imperial College of Science and Technology.

Dr. C. CHREE, F.R.S., President, in the Chair.

A Special General Meeting was held at which the following Resolution was confirmed:—

*Resolution*:—"That the Articles of Association contained in the printed document submitted to the Meeting, and for the

purpose of identification, subscribed by the Chairman thereof, be, and the same are hereby approved; and that such Articles of Association be, and they are hereby, adopted as the Articles of Association of the Society."

An Ordinary Meeting of the Society was then held.

The following Papers were read:—

1. Experiments on a Directive System of Wireless Telegraphy.  
By MESSRS. E. BELLINI and A. TOSI.
2. The Lateral Vibration and Deflection of Clamped-Directed Bars. By Dr. J. MORROW.
3. The Resistance of a Conductor of Uniform Thickness whose breadth suddenly changes, and on the shape of the Stream-lines.  
By Prof. C. H. LEES.
4. The Inductance of two Parallel Wires. By Dr. J. W. NICHOLSON.
5. Homogeneous Secondary Radiation. By Dr. C. G. BARKLA and Mr. SALLER.
6. Note on the amount of Water in a Cloud formed by expansion of moist air. By Prof. W. B. MORTON.
7. An Elementary Treatment of the Motion of a Charged Particle in a Combined Electric and Magnetic Field. By Prof. W. B. MORTON.

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October 23rd, 1908.

Meeting (informal) held at the National Physical Laboratory, by the kind invitation of the Director, Dr. R. T. GLAZEBROOK, F.R.S.

The laboratories were thrown open for inspection, and demonstrations were given by Mr. Bairstow, Mr. Campbell, Dr. Harker, Mr. Jeffcott, Mr. Paterson, Dr. Rayner, Mr. Rosenhain, Mr. Seaton, and Mr. Stanton.

A large number of Fellows and Visitors were present.

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November 13th, 1908.

Meeting held at University College by the kind invitation  
of Prof. F. T. TROUTON.

Dr C. CHREE, F.R.S., President, in the Chair.

The following Candidate was elected a Fellow of the Society :—

B. J. PALMER.

The following resolution of condolence on the death of Prof. W. E. Ayrton was moved by the President, supported by Prof. Carey Foster and Prof. S. P. Thompson, and unanimously adopted :—

“This Meeting of the Physical Society desires to record its sense of the great loss sustained by Physical Science and by the Society through the death of their Past-President, Professor Ayrton, and to express their sincere sympathy with Mrs. Ayrton and the other members of Professor Ayrton's family in their bereavement.”

The following Papers were read :—

1. The Photo-electric Properties of Potassium-sodium Alloys.  
By Prof. J. A. FLEMING.
2. The Recombination of Ions in Air. By Dr. P. PHILLIPS.
3. The Influence of a Magnetic Field on the Photographic Patterns produced by the Electric Spark. By Prof. A. W. PORTER.
4. The Rate of Growth of Viscosity in Congealing Solutions.  
By Mr. A. O. RANKINE.
5. The Isoneres of Adsorption by Solid Surfaces from Solutions.  
By Prof. F. T. TROUTON.

Prof. A. W. PORTER showed an Experiment illustrating the Anomalous Effect of Lagging thin Wires.

November 27th, 1908.

Meeting held at the Imperial College of Science and Technology.

Dr. C. CHREE, F.R.S., President, in the Chair.

The following Candidate was elected a Fellow of the Society :—

R. L. JONES.

The following Papers were read :—

1. A Graphic Method of Dealing with Refracting Surfaces.  
By Mr. H. S. ALLEN.

2. An Accurate Method of Measuring Moments of Inertia. By the late Prof. W. CASSIE.
3. The Diffusion of Actinium and Thorium Emanations. By Mr. S. RUSS.
4. Elliptic Polarization. By Mr. J. WALKER.
5. An Experimental Investigation of Gibbs's Theory of Surface-Concentration. By Mr. W. C. M. LEWIS.

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December 11th, 1908.

Meeting (informal) held at the Imperial College of Science and Technology.

An Exhibition of Apparatus was given by the following firms:—Messrs. The Cambridge Scientific Instrument Co.; C. F. Casella & Co.; A. C. Cossor Ltd.; G. Cussons Ltd.; J. H. Dallmeyer Ltd.; Elliott Bros.; Everett, Edgumbe & Co. Ltd.; A. Gallenkamp & Co. Ltd.; Gambrell Bros.; J. J. Griffin & Sons; A. Hilger Ltd.; India Rubber, Gutta Percha & Telegraph Works Co., Ltd.; Hans Knudsen; Marconi's Wireless Telegraph Co. Ltd.; Nalder Bros. & Thompson Ltd.; Newton & Co.; R. W. Paul; Pitkin & Co.; Siemens Bros. & Co. Ltd.; Snell & Tinsley; Strange & Graham Ltd.; Swift & Son; Alexander Wright & Co. Ltd.; Carl Zeiss Ltd.

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January 22nd, 1909.

Meeting held at the Imperial College of Science and Technology.

Dr. C. CHREE, F.R.S., President, in the Chair.

The following Candidates were elected Fellows of the Society:—

H. BANISTER, H. BRUNSKILL, F. P. FULLER, T. HARRIS,  
W. WILLINGS.

The following Papers were read:—

1. The Effective Resistance and Inductance of a Concentric Main. By Dr. A. RUSSELL.
  2. Luminous Efficiency of a Black Body. By Dr. C. V. DRYSDALE.
  3. The Use of the Potentiometer on Alternate Current Circuits. By Dr. C. V. DRYSDALE.
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*Annual General Meeting.*

February 12th, 1909.

Meeting held at the Imperial College of Science and Technology.

Dr. C. CHREE, F.R.S., President, in the Chair.

The following Report of the Council was read by the Secretary:—

SINCE the last Annual General Meeting, ten ordinary Science Meetings, two informal Meetings, and two Special General Meetings of the Society have been held. Of these, eleven were held at the Royal College of Science; one, that on March 27th, at the Northampton Institute, by invitation of Dr. R. M. Walmsley; one, that on Oct. 23rd, at the National Physical Laboratory, by invitation of Dr. R. T. Glazebrook, F.R.S.; and one, that on Nov. 13th, at University College, by invitation of Prof. F. T. Trouton, F.R.S. The average attendance at the Meetings has been 61 as compared with 58 during the preceding year.

The Fourth Annual Exhibition of Apparatus by Manufacturers was held on December 11th, when the attendance of Fellows and Visitors amounted to about 300. Greater advantage is now being taken of this event by exhibitors, and the attendance this year was much larger than hitherto.

Certain alterations to the Articles of Association have been introduced, more particularly in regard to the election of the Council. A copy of the revised Articles has been forwarded to every Fellow of the Society.

The number of Ordinary Fellows now on the roll, as distinct from Honorary Fellows, is 430, a decrease of 7 on the number last year; 12 new Fellows have been elected. There have been five resignations; the names of three Fellows have been struck off the List for non-payment of subscriptions; and the Society has to mourn the loss by death of two Honorary Fellows, namely, H. Becquerel and E. Mascart; also one of their Past Presidents, Prof. W. E. Ayrton, one of their Secretaries, Prof. W. Cassie, and nine other Ordinary Fellows, namely, Lord Blythwood, Mr. S. Hall, Prof. A. S. Herschel, Mr. S. Joyce, Mr. F. J. M. Page, the Earl of Rosse, Capt. J. H. Thomson, Mr. J. F. Walker, and Dr. W. E. Wilson.



The Report of the Council was adopted.

The Report of the Treasurer and the Balance Sheet were presented and adopted.

The election of Officers and other Members of Council then took place, the new Council being constituted as follows :—

*President.*—C. CHREE, Sc.D., LL.D., F.R.S.

*Vice-Presidents who have filled the Office of President.*—Prof. G. C. FOSTER, F.R.S.; Prof. W. G. ADAMS, M.A., F.R.S.; Prof. R. B. CLIFTON, M.A., F.R.S.; Prof. A. W. REINOLD, M.A., F.R.S.; Sir ARTHUR W. RÜCKER, M.A., D.Sc., F.R.S.; Sir W. DE W. ABNEY, R.E., K.C.B., D.C.L., F.R.S.; SHELFORD BIDWELL, M.A., LL.B., F.R.S.; Prin. Sir OLIVER J. LODGE, D.Sc., F.R.S.; Prof. S. P. THOMPSON, D.Sc., F.R.S.; R. T. GLAZEBROOK, D.Sc., F.R.S.; Prof. J. H. POYNTING, M.A., D.Sc., F.R.S.; Prof. J. PERRY, D.Sc., F.R.S.

*Vice-Presidents.*—W. DUDELL, F.R.S.; Prof. A. SCHUSTER, Ph.D., F.R.S.; S. SKINNER, M.A.; W. WATSON, D.Sc., F.R.S.

*Secretaries.*—W. R. COOPER, M.A.; S. W. J. SMITH, M.A., D.Sc.

*Foreign Secretary.*—Prof. S. P. THOMPSON, D.Sc., F.R.S.

*Treasurer.*—Prof. H. L. CALLENDAR, M.A., LL.D., F.R.S.

*Librarian.*—W. WATSON, D.Sc., F.R.S.

*Other Members of Council.*—A. CAMPBELL, B.A.; W. H. ECCLES, D.Sc.; A. GRIFFITHS, D.Sc.; J. A. HARKER, D.Sc.; Prof. C. H. LEES, D.Sc., F.R.S.; T. MATHER, F.R.S.; A. RUSSELL, M.A., D.Sc.; Prof. E. RUTHERFORD, D.Sc., F.R.S.; F. E. SMITH; R. S. WHIPPLE.

Votes of thanks were passed to the Auditors, to the Officers and Council, and to the Governors of the Imperial College of Science and Technology.

The President then delivered an Address.

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## TREASURER'S REPORT.

AFTER the temporary boom of 1907 has succeeded another period of depression. It has been almost impossible, in spite of strenuous efforts, to collect any arrears of subscriptions, the amount under this head standing at £4 4s. 0d. as compared with £54 12s. 0d. in 1907. Only nine new Fellows have paid entrance fees as compared with 27 in 1907, and the amount from composition fees is unusually small. In other respects there is little change in the position or expenditure of the Society, except that last year's Printing Bill was unusually heavy, and the balance, allowing for liabilities, is slightly reduced in consequence. The provision of refreshments at the Meetings has entailed a small additional outlay, but the catering has been carried out most economically, and the total expense for 1908 is little more than that for the exhibition of apparatus alone in 1907.

HUGH L. CALLENDAR,

*Hon. Treasurer.*



# PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, DECEMBER 31, 1908.

ASSETS.		LIABILITIES.	
	£ s. d.		£ s. d.
Subscriptions due, Treasurer's estimate .....	70 0 0	Printing .....	94 13 1
£553 Furness Ry. Co. 3 per cent. Debenture Stock .	405 0 0	Cheque not presented .....	2 2 0
£1600 Midland Railway 2½ per cent. Preference Stock .....	1104 0 0		
£200 Metropolitan Board of Works 3½ per cent. Consolidated Stock .....	205 0 0		
£400 Lancaster Corporation 3 per cent. Redeemable Stock .....	332 0 0		
£284 2s. 9d. New South Wales 3½ per cent. Inscribed Stock .....	252 15 0		
£500 London, Brighton, and South Coast Railway Ordinary Stock .....	530 0 0		
£500 Great Eastern Railway 4 per cent. Debenture Stock .....	555 0 0		
Balance in Bank .....	56 6 6		
On deposit .....	400 0 0		
Publications in Stock, estimated .....	200 0 0	Balance .....	4013 6 5
	<u>£4110 1 6</u>		<u>£4110 1 6</u>

Audited and found correct,

HUGH L. CALLENDAR, *Honorary Treasurer.*

21st January 1909.

HENRY R. J. BURSTALL, } *Auditors.*  
A. A. C. SWINTON, }



*Presidential Address to THE PHYSICAL SOCIETY OF LONDON,*  
*12th February, 1909.*

C. CHREE, ScD., LL.D., F.R.S.

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§ 1. During the past year our losses have been heavy. In Prof. Ayrton we have lost an ex-president whose services to technical education have been widely recognised. Several notices of Prof. Ayrton have appeared by those whose personal knowledge of him was much closer than mine; in particular I would refer to the article by Prof. Perry in "Nature." Prof. Ayrton was a not-infrequent attendant at meetings of the Society, and when he came he had usually something interesting to say on the papers before the meeting. Prof. Cassie, our late Secretary, was a personal friend of my own, our acquaintance extending back to 1882 when we were undergraduates at Cambridge. Prof. Cassie graduated at Aberdeen University in 1881, taking the first place in the examination for honours in Mathematics and Natural Philosophy (Physics). He had the distinction of winning the Ferguson Scholarship, open to graduates of the four Scottish Universities. At Cambridge he was 5th Wrangler and obtained a 1st Division, 1st Class, in the final part of the Mathematical Tripos. He was also the first to be elected to the Clerk-Maxwell Studentship. After engaging for some time in University Extension lecturing and private tuition, he was appointed to the Professorship of Physics at the Royal Holloway College, previously held by Prof. Callendar. His energy and efficiency when acting as our Secretary will I think be recognised by all who came in contact with him in that capacity.

Professor Cassie's death was exceptionally sudden, and happening when it did at the end of the Spring Session, it might have led to confusion in the secretarial work, but for the public spirited action of Mr. H. M. Elder, who put his long experience as Secretary at the disposal of the Council. Mr. Elder continued to act as Secretary until the appointment in November of Dr. S. W. J. Smith.

§ 2. During the year the Council passed in review the Articles of Association, and a variety of small changes were introduced, which were approved by statutory meetings of the Society.



Probably the most important change related to procedure intended to prevent Fellows' annual contributions falling into arrear. Our Treasurer had found that even his powers of persuasion did not invariably meet with the success which they deserved.

As has now been the practice for some years, an exhibition of instruments was held in the Autumn Session. Whether regarded from the number and interest of the exhibits, or from the number of Fellows and Visitors who attended, the exhibition was eminently successful. For this the Society is mainly indebted to our Senior Secretary, Mr. W. R. Cooper, and to the active co-operation of Professor Callendar and the Staff of the Physics Department of the Imperial College of Science and Technology.

§ 3. As you have only just confirmed the election of two new Honorary Fellows, it may perhaps appear premature to refer to the subject. Unless, however, the matter is referred to to-night, it may be overlooked. I am afraid that eminent unquestionably as all our Honorary Fellows are, it is not every Fellow, perhaps not even every member of the Council, who would welcome a *viva voce* examination as to the exact sources of their distinction. Of Prof. Julius Thomsen I must confess my knowledge is mainly second-hand, as his subject, physical chemistry, is one with which I can hardly claim an intimate personal acquaintance. His contributions to thermo-chemistry are, however, very numerous, and his reputation in that domain is world wide. Dr. René Benoit is the Director of the International Bureau of Weights and Measures at Sèvres. The work of that institution relates primarily to the standards of length and mass, but the study of standards of length entails minute study of temperature, and has thus brought thermometry within the domain of the Bureau International. From the point of view of accuracy, the work at Sèvres has probably never been excelled. The Committee in supreme control of the Bureau International contains representatives from all the leading scientific countries, and though these gentlemen are usually selected principally with reference to their official position, their number usually includes several of the most capable metrologists of the day. The staff at Sèvres has also included in addition to Dr. Benoit other eminent physicists, notably Dr. Guillaume and Dr. Chappuis. Thus the excellence of the work done at Sèvres must be ascribed to various sources, but much is due unquestionably to Dr. Benoit himself, to his skill as an observer and his insight in discussing experimental results.

§ 4. Leaving now historical matters, I propose giving an account of some work in which I have recently been engaged. Last year I described some of the phenomena observed in the regular diurnal variation of Terrestrial Magnetism, dealing more particularly with the results of the National Antarctic Expedition of 1902-4. To-night I intend to discuss some phenomena relating to the irregular variations in Terrestrial Magnetism.

When the National Antarctic Expedition set out, one of the items in the programme was a certain scheme of observations in which it was hoped that a large number of observatories would co-operate. According to this scheme, which originated I believe in Germany, there were two pre-arranged term-days a month, throughout which hourly readings at exact hours G. M. T. should be taken, and in each term-day there was a pre-arranged term-hour, during which it was hoped that the drums of magnetographs would be rotated at twelve times the usual speed, so as to secure a very open time scale. The principal object, presumably, was to facilitate the intercomparison of the records of the disturbances, of which in the usual course a considerable number would naturally present themselves during the term-hours.

For the comparison of the records from different stations of disturbances of considerable size an open time scale is most valuable, as the times of occurrence of individual phenomena can be determined with so much higher precision than in the ordinary slow-run curves. When, however, there are no large disturbances, quick-run curves lose their advantage to a large extent, and in some respects they are even inferior to ordinary curves. A difference of 1 mm. in the ordinates of two points catches the eye much more readily when their difference of abscissa is 1 cm. than when it is 12 cm. 1902 and 1903, especially the former, were years of few sun-spots, and as is usually the case near sun-spot minimum magnetic disturbances were relatively few, and with one or two exceptions of minor size. As it so happened, the selected term-days were all quiet days, and there was not a single term-hour in which any but the most trifling disturbance presented itself at the co-operating stations.

§ 5. The term-days having thus failed to supply material for a study of disturbances, I made, with the sanction of the Royal Society's Magnetic Committee, an addition to the pre-arranged programme. A list was prepared of the days of largest magnetic disturbance in 1902 and 1903 for which Antarctic records existed, and this

was sent to the directors of the observatories at Christchurch (New Zealand), Mauritius, and Colaba (Bombay), with the request that they would supply copies of their magnetic curves for these days. To the three directors of the above mentioned observatories, Dr. Coleridge Farr, Mr. C. T. F. Claxton, and Mr. N. A. F. Moos, I am much indebted for their generous response to the request. I am similarly indebted to Mr. E. Kitto and the Committee of the Falmouth Observatory for the loan of a number of Falmouth magnetic curves. Even in 1902 the Kew records suffered perceptibly from electric tram currents, so that records from a second and undisturbed English Station were highly useful. The results to which reference will be made to-night are derived from the Christchurch, Mauritius, Colaba, Falmouth and Kew records, and from those at the 'Discovery's' Winter Quarters in the Antarctic (lat.  $77^{\circ} 51'$  S., long.  $166^{\circ} 37'$  E.). The full discussion will form part of the volume still to be published on the Antarctic magnetic results. For permission to anticipate to-night to some extent the publication of that volume I am indebted to the President of the Royal Society and the Director of the National Physical Laboratory.

§ 6. Immediately on the discovery by Prof. Hale of the Zeeman effect in light from sun-spot areas, indicating an intense magnetic field, several astronomers suggested that the mystery of magnetic storms had received its final solution. They seemed unaware of the fact that magneticians had not waited for any such visible sign before endowing the Sun with electric currents and magnetic fields, and that the possibilities of direct magnetic action exerted by the Sun had been considered by a succession of physicists. It is many years since it was recognised that the size of magnetic disturbances constituted a serious difficulty. Lord Kelvin, for instance, made a rough estimate of the expenditure of energy which such direct action would necessitate during a large magnetic storm, and came to the conclusion that it was inadmissibly large. A few weeks ago, Dr. A. Schuster published in "Nature" an estimate of the possible size of the effect on the Earth of the largest of sun-spot areas, endowed with what he considered the largest magnetic moment reasonably attributable, and found it practically negligible.

Some years ago Mr. E. W. Maunder in an important series of papers advanced evidence from a list of Greenwich magnetic storms that these tend to follow one another at intervals of about twenty-seven and a quarter days, answering to the revolution

period of solar meridians relative to the earth. He postulated the ejection from spot areas on the sun of some kind of discharge, travelling like a water jet, which on reaching the Earth set up magnetic storms, the discharge from a single area of the Sun persisting sometimes for several rotation periods. Mr. Maunder suggested that as the discharge formed only a narrow stream, this removed the difficulty raised by Lord Kelvin on the ground of the large expenditure of energy required. As I pointed out, however, this apparently overlooks the fact that on Mr. Maunder's theory the disturbance on the Sun continues for many days or even months, while Kelvin supposed it to last only while the magnetic storm was experienced on the Earth. Further, as severe magnetic storms usually last for many hours, sometimes for several days, the cross section of one of Mr. Maunder's hypothetical jets must be very considerable. It is thus by no means obvious that his theory postulates less expenditure of energy than Kelvin's estimate. How much energy it entails must depend on the nature of the solar discharge, and as to that Mr. Maunder makes no definite suggestion. The discharge might be electrified particles, as Arrhenius has suggested, Röntgen rays, as Nordmann postulates, or kathode rays, as Prof. Birkeland advocates, or some entity not yet recognised.

§ 7. If magnetic storms are due to a discharge of particles or ions travelling from the Sun, the paths of these when near the Earth are largely determined by the Earth's magnetic field. According to Störmer's elaborate mathematical calculations, the electric carriers will be confined to spaces near the Earth's magnetic poles. The law of variation of the resulting disturbances at different parts of the Earth's surface will naturally be very complicated. All that can very well be inferred is that magnetic disturbances will be less in equatorial regions than elsewhere. If, on the other hand, the Sun behaves as a magnet, one would expect the phenomena to be relatively simple. The distance of the Earth from the Sun is so large compared to the Sun's diameter, and still more so when compared to the Earth's diameter, that a magnetic force originating in the Sun would be practically constant throughout the Earth and its atmosphere if the Earth itself were non-magnetic. The Earth's magnetism of course is a complication, but according to the Gaussian theory the magnetism is internal, and in the expression for the potential terms of higher order are small compared to those of the first. It is thus natural to suppose that

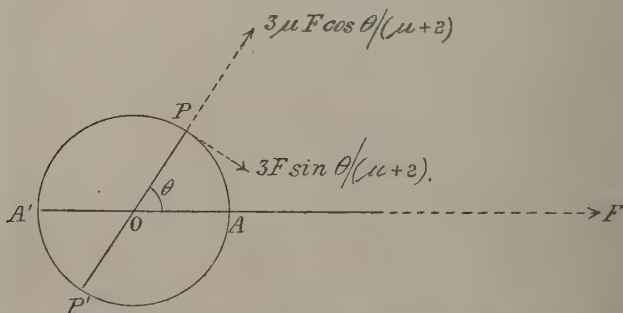
magnetically the Earth's material must in the main be fairly uniform, or that the magnetic quality may to a first approximation be treated as a function only of the distance from the centre.

In the present state of our knowledge I do not think we are justified in regarding direct action from the Sun as an impossibility. We do find that in at least some storms magnetic changes of similar size and character present themselves at very distant stations, and in some instances they arise suddenly at times of profound quiet. The phenomena unquestionably do suggest the entrance of the Earth into a stray magnetic field. It is quite conceivable, as I have remarked before, that such fields might arise from some source quite other than the Sun. It seems thus worth while considering what the consequences would be of the existence of an external field, uniform but for the presence of the Earth.

§ 8. If we regard the Earth as a sphere of radius  $a$ , having uniform permeability  $\mu$ , and if  $F$  denote the strength which the disturbing field would possess if the Earth were not magnetic, the potential  $V$  is given by

$$V = -Fr \cos \theta + \frac{\mu - 1}{\mu + 2} F \frac{a^3}{r^2} \cos \theta. \quad \dots \quad (1)$$

Here  $r$  and  $\theta$  are polar co-ordinates, the origin being at the Earth's centre, and the initial line  $\theta = 0$  answering to the direction of the field  $F$ .



In the figure,  $A P A P'$  is a section through  $O$  the centre of the sphere,  $A' A$  being the direction of the impressed field  $F$ .

The force at any point  $P (a, \theta)$  just outside the sphere may be regarded as composed of

a normal component  $3\mu F \cos \theta \div (\mu + 2)$ ,  
and a tangential component  $3F \sin \theta \div (\mu + 2)$ .



If  $\psi$  be the inclination to OA of the resultant force R at P, then

$$\tan \psi = (\mu - 1) \sin 2\theta \div \{(\mu + 1) + (\mu - 1) \cos 2\theta\}, \quad \dots \quad (2)$$

and

$$R/F = \{3/(\mu + 2)\} \{\mu^2 - (\mu^2 - 1) \sin^2 \theta\}^{\frac{1}{2}} = \{3/(\mu + 2)\} \{1 + (\mu^2 - 1) \cos^2 \theta\}^{\frac{1}{2}}. \quad (3)$$

If  $(\mu - 1)/1$  be small, then  $\psi$  is necessarily small, and  $R/F$  can differ but little from unity.

Over the surface, R obviously has its maximum value where  $\theta = 0$ , and its minimum where  $\theta = \pi/2$ , and

$$\left. \begin{aligned} R_{\max.}/F &= 3\mu/(\mu + 2), \\ R_{\min.}/F &= 3/(\mu + 2) \end{aligned} \right\}, \quad \dots \quad (4)$$

so that

$$R_{\max.}/R_{\min.} = \mu. \quad \dots \quad (5)$$

The maximum value occurs at the ends of the diameter  $\theta = 0$ , where the force is entirely vertical, the minimum at all points on the perpendicular great circle, where the force is entirely tangential (*i.e.* horizontal).

Clearly, everything turns on the size of  $\mu$ . If  $\mu$  were large, R would have a wide range of values, and the disturbing forces at different parts of the surface would vary widely in direction. It is, however, difficult to believe that  $\mu$  is large. For one thing, the earth's field itself is only of the order 0.5 C.G.S. and a disturbance as large as .01 C.G.S. is very exceptional. Even in the best magnetic steel  $\mu$  is low in small fields. Apart from mere theoretical considerations, there is the fact that if  $\mu$  were large the horizontal (*i.e.* tangential) component  $3F \sin \theta/(\mu + 2)$  would tend to be negligible compared to the vertical  $3\mu F \cos \theta/(\mu + 2)$  except at places for which  $\theta$  is nearly  $\pi/2$ . Now the tendency is all the other way. Vertical force curves at most places are conspicuously **quieter than horizontal force curves**.

A point worthy of special remark is that R has the same value, and the same absolute direction in space, for places whose angular co-ordinates are  $\theta$  and  $\pi + \theta$ ; *i.e.*, the disturbing forces are identical at any two places on the surface which are diametrically opposite to one another.

§9. As the Earth's crust is on the whole non-magnetic, it is clear that the above simple case cannot be an exact representation of the facts. It may thus be worth while glancing at the next simplest hypothetical case, in which we have an earth consisting

of a nucleus of radius  $a$  and permeability  $\mu$ , with a surrounding crust of lesser permeability  $\mu'$  and external radius  $a'$ . In this case, supposing the impressed field of strength  $F$  as before, we find for the potential external to the Earth

$$V = -Fr \cos \theta + F \frac{a'^3}{r^2} \cos \theta \frac{(\mu + 2\mu')(\mu' - 1) + (\mu - \mu')(2\mu' + 1)(a/a')^3}{(\mu + \mu' 2)(\mu' + 2) + 2(\mu - \mu')(\mu' - 1)(a/a')^3} \quad (6)$$

The only sub-case of practical interest is that in which  $\mu' - 1$  is small, *i. e.* in which the material of the surface layer or crust is only slightly magnetic. In this case, assuming  $a/a'$  not to be very small, we find the potential approximately given by

$$V = -Fr \cos \theta + F\{(\mu - 1)/(\mu + 2)\}a^3 \cos \theta / r^2. \quad (7)$$

This is identical in form with (1), only whilst  $a$  in (1) represents the Earth's radius, in (7) it represents the radius of the highly magnetic nucleus. If the external slightly magnetic layer be thin, the phenomena are much the same as in the case of uniform permeability  $\mu$ . If  $(a' - a)/a'$  be considerable, the variation of  $R$  over the surface is materially reduced. Whether  $(a' - a)/a'$  be small or not, the values of  $R$  at diametral points are necessarily equal.

§10. Before dealing with the magnetic curves, there is one point I had better explain. The most usual procedure is that followed at Kew, Falmouth, and Colaba, where movement of the photographic trace up the sheet signifies increase in the magnetic elements, declination, horizontal force and vertical force. At Mauritius, however, movement up the sheet answers to increase only in the declination; in the case of the horizontal and vertical forces it signifies decrease. At Christchurch movement up the sheet denotes decrease in all three elements. In the Antarctic curves it denotes increase for the vertical force, but decrease for the two other elements. As the question naturally suggests itself why the original curves were not inverted when necessary during the reproduction, so as to secure that in all movement up the sheet should answer to increase in the element considered, I had better mention the reasons for not doing so. The three elements in the Antarctic were all recorded on one sheet, and to have inverted the declination and horizontal force traces, while leaving the vertical force traces unchanged, would have been troublesome. Then, it must be remembered that increase and decrease in declination are purely relative terms. Declination is westerly at Kew, Falmouth, and Mauritius, but easterly at Colaba and Christchurch. Thus increase of declination at the two last-



mentioned stations means exactly the opposite of what it does at the first three. At the 'Discovery's' Winter Quarters declination was easterly, but about  $152\frac{1}{2}^{\circ}$ —i. e. the north pole of the magnet pointed about  $27\frac{1}{2}^{\circ}$  to the east of south—thus an increase of declination answered to a westerly directed disturbing force.

At Kew an increase in the horizontal force  $H$  means a disturbing force inclined at  $16\frac{1}{2}^{\circ}$  to the west of north; at Christchurch it means a force inclined at  $16^{\circ}$  to the east of north; in the Antarctic it means a force inclined at  $27\frac{1}{2}^{\circ}$  to the east of south. Then at Winter Quarters and Christchurch, where the longitude is nearly  $180^{\circ}$  E., the west has nearly the same absolute direction in space as has the east at Kew and Falmouth.

Finally, increase in the vertical force means at Kew, Falmouth, and Colaba a force directing the north pole of a magnet towards the Earth's centre, whereas at Mauritius, Christchurch, and Winter Quarters it means a force outwardly directed. Taking these points into consideration, I decided to invert none of the curves.

At Falmouth, Colaba, Mauritius, and Christchurch, the magnetographs are of the Kew pattern, with nearly equal time scales. The sensitiveness is also nearly the same for declination and horizontal force at the four stations. This greatly facilitates intercomparison of the curves.

§11. I now pass to some lantern-slides illustrating various magnetic disturbances recorded in 1902 and 1903. The first slide (fig. 1) shows the records of horizontal force on April 5-6, 1903, at Falmouth, Colaba, Mauritius, and Christchurch, with the corresponding declination records from the first and last of these stations. The four central curves are the horizontal force ones. Corresponding times (G.M.T.) at the various stations come at least very approximately in the same vertical line. For some hours previous to 11 p.m. on April 5 the curves were very quiet. Then suddenly the horizontal force trace begins to go up the sheet at Falmouth and Colaba, and down the sheet at Mauritius and Christchurch, thus indicating an increase of force at all four stations. The time scale of the original curves\* is about 1 hour to 15 m.m., thus, rapid as the initial change of force is compared to the ordinary standard of magnetic changes, it is by no means instantaneous. It occupied in fact about 4 minutes.

With magnetic curves of the type, time can hardly be measured to nearer than 1 minute, so that all one can say is that the

\* The curves are all shown here on a reduced scale.

initial movements appeared to commence simultaneously at about 11.25 P.M., G.M.T., on April 5. Such sudden movements are a fairly common prelude to large magnetic disturbances, and at a particular station they usually have a characteristic type. Thus at Kew, the initial change of horizontal force is usually an increase. In the present case the initial movement though not peculiar to the horizontal force is less marked in the declination. Though the Colaba and Mauritius horizontal force curves are, as explained, inverted relative to one another, their close resemblance will be readily recognised.

Though there is a fairly close general resemblance between later parts of the records from the different stations, the difference in details renders it difficult to identify corresponding movements. Even at the same station the turning points in the curves of the different elements did not appear to occur simultaneously.

The storm of April 5-6 was hardly worthy of so conspicuous an initial movement. At Kew and Falmouth the total range of declination was only about 25'.

§ 12. As already stated, the curves at the co-operating stations were very quiet for some hours prior to 11 P.M. on April 5, but earlier in the day between 11 A.M. and 5 P.M. there were a few rapid though small movements.

As our next slide (fig. 2) shows, these earlier disturbances synchronised with very much larger disturbances in the Antarctic. To bring all the more striking of the Antarctic disturbances of April 5-6 within the compass of one figure, five or six hours trace have been omitted, being represented only by a small gap. It is the portion subsequent to the gap that answers to the disturbances shown in fig. 1. It should be noticed that whilst Greenwich time is used in fig. 1 and the other illustrations of disturbances at the co-operating stations, the time employed in all the Antarctic curves is the local time (L.T.) of Winter Quarters, which was 11 hours 7 minutes fast on Greenwich.

In fig. 2 and the other reproductions of Antarctic curves the letters D, H, V denote the declination, horizontal force, and vertical force traces,  $D_0$ ,  $H_0$ ,  $V_0$  the corresponding base lines. T denotes the temperature inside the vertical force box. The scales and base-line values are shown at the margins.

The Antarctic curves were practically never quiet, so it is difficult to assign a definite beginning or end to a disturbance

Fig. 1.  
April 5-6, 1903.

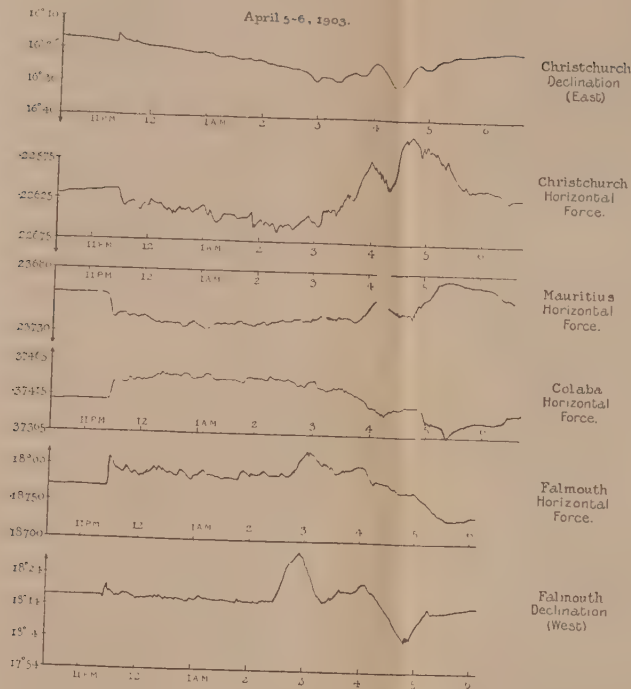


Fig. 2.—Antarctic, April 6, 1903.

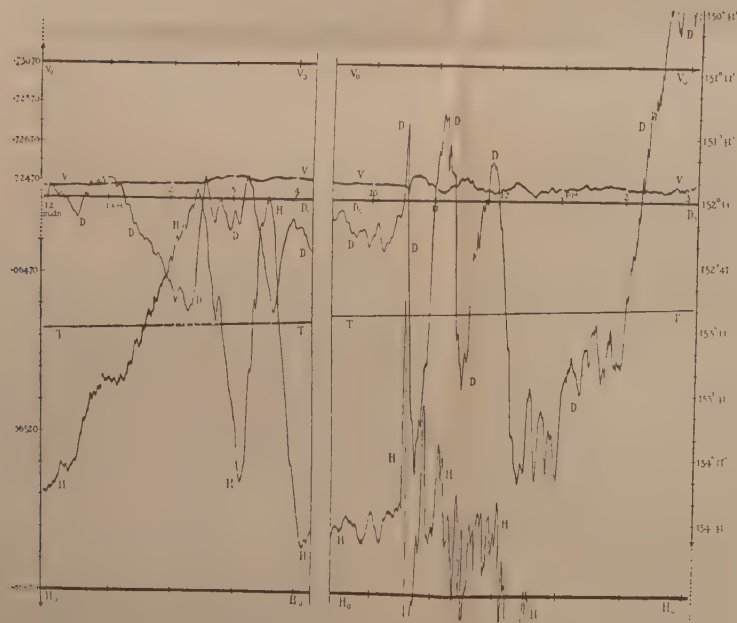


Fig. 3.

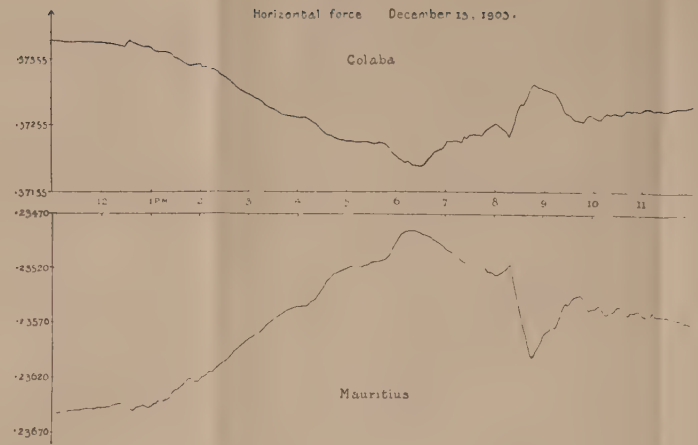


Fig. 4.—Antarctic, December 13-14, 1903.

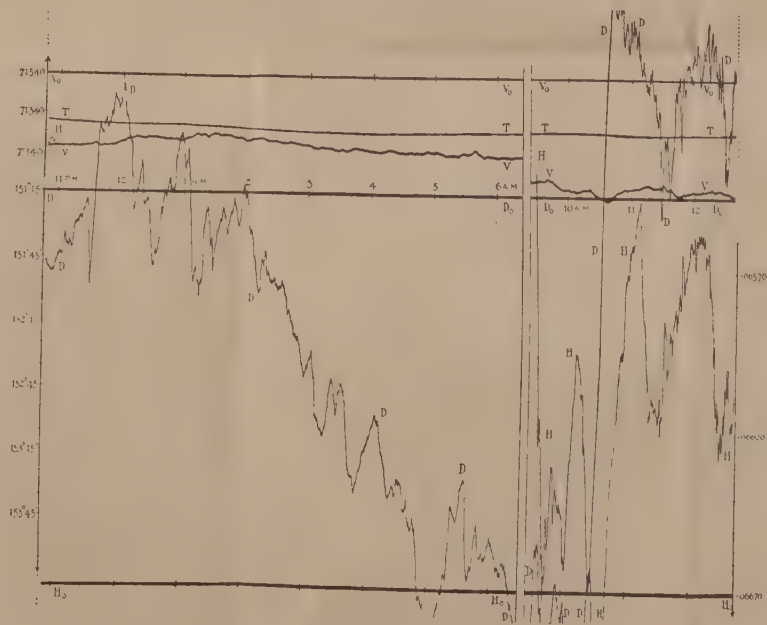


Fig. 5.

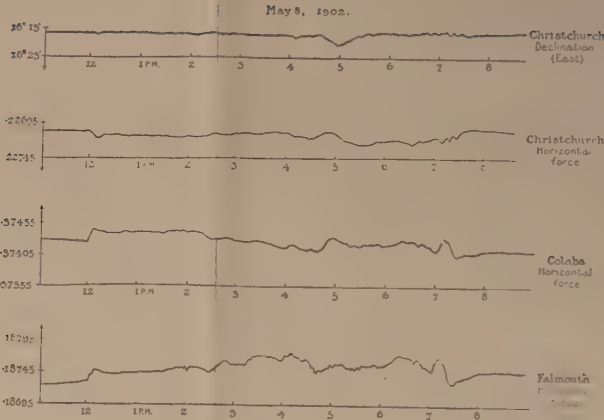
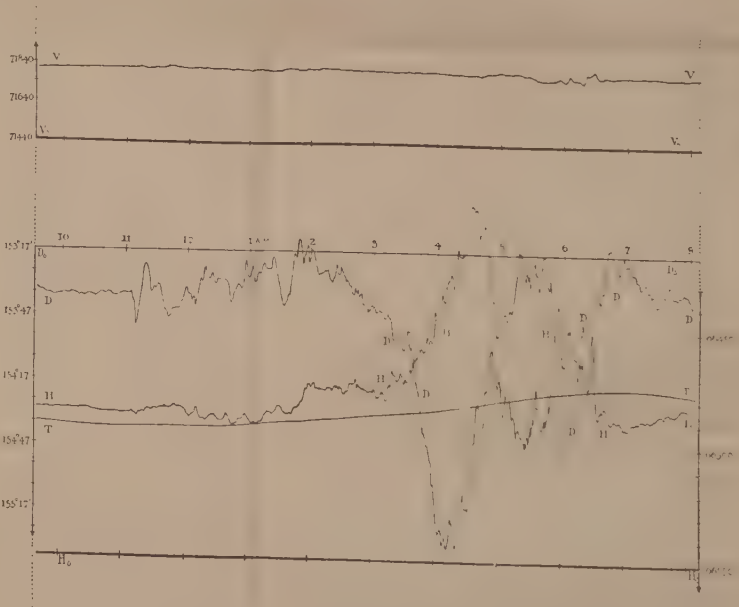


Fig. 6.—Antarctic, May 8-9, 1902.





there. What answers to the sudden commencement at the co-operating stations on April 5 is apparently the large sudden movement a little to the right of the gap in fig. 2. Commencing at the same time, 11.25 P.M. G.M.T. on the 5th (10.32 A.M. L.T. on the 6th) it also lasted 4 minutes. But whilst at the co-operating stations the initial movement was much more prominent than the movements immediately following it, the converse was true of the Antarctic. The initial movement there was immediately followed by an equally rapid and notably larger movement in the opposite direction. A further peculiarity of the Antarctic was that the initial movement was a *decrease* in all three elements.

§ 13. In this instance I have calculated, from the curve measurements, the components relative to 3 fixed axes of the disturbing force to which the initial movement may be ascribed. The axes were taken one parallel to the Earth's axis of rotation, and the others perpendicular, one being in the meridian containing Kew (or to the present degree of approximation Greenwich) and the other in the meridian  $90^\circ$  east. The north being taken as the initial direction, the inclinations to it of the disturbance vector were found to be  $68^\circ$  at Kew,  $61^\circ$  at Winter Quarters,  $40^\circ$  at Christchurch, and  $11^\circ$  at Colaba. The azimuths of the meridian planes containing these vectors were  $224^\circ$  E. at Kew,  $189^\circ$  E. at Winter Quarters,  $167^\circ$  E. at Colaba, and  $148^\circ$  E. at Christchurch. There is no very great degree of accuracy possible in the measurements of the disturbances, as they are small when considered absolutely; thus the departure from parallelism between the vectors at the several stations is not very striking. On the whole these results are by no means unfavourable to the hypothesis of direct solar action. The fact that the vertical force nearly vanished at Christchurch, where the disturbing force was least, is distinctly favourable; but some of the phenomena point to the opposite conclusion. The values found for the resultant disturbing force were  $23 \gamma$  at Christchurch,  $27 \gamma$  at Colaba, and  $45 \gamma$  at Kew, but  $102 \gamma$  at Winter Quarters. On the hypothesis of direct solar action this would imply a value of  $\mu$  greater than 4.

Further, it will be remembered that on that hypothesis the vertical force ought to be largely in the ascendant at stations where the disturbing force is largest. But at Winter Quarters the vertical component was less than a third of the horizontal.

§ 14. The next few slides afford further examples of suddenly

commencing disturbances. The first (not reproduced) shows the declination and horizontal force disturbances at Falmouth on December 13, 1903. The main disturbance on this date exceeded that on April 5-6, 1903, but though it also possessed a sudden commencement, the initial movement was of a somewhat confused character. The next slide (fig. 3) shows the horizontal force changes at Colaba and Mauritius on December 13, 1903, corresponding to those at Falmouth just referred to. The initial movement, an increase of force—increase in the Mauritius curve answering to movement down the sheet—was much more conspicuous than at Falmouth. The initial rise was followed by an almost continual fall continuing for about 5 hours, the total decrease at either station amounting to nearly 200  $\gamma$ .

In the Antarctic, as the next slide (fig. 4) shows, December 13 was highly disturbed. To bring the main movements within the limits of one figure, about three hours trace are omitted between 6 and 10 A.M. L.T. on the 14th. The sharp oscillation in the declination trace between 11 P.M. and midnight L.T. on the 13th answers to the commencing movement in fig. 3. The largest and most sudden movements, however, are those shown on the 14th, subsequent to the gap in fig. 4. The declination trace will be seen to traverse the full width of the sheet, representing nearly  $5^\circ$ , in about 20 minutes. It did not, however, prove possible to identify any of these large Antarctic movements with movements elsewhere.

§ 15. The next slide (not reproduced) shows sudden commencing movements in horizontal force which occurred on August 20 and November 6, 1902, at the co-operating stations. At Winter Quarters on August 20, 1902, the declination shows a sudden change lasting 4 minutes and apparently synchronous with the sudden commencement elsewhere; but it was immediately followed, as on April 5, 1903, by an equally rapid and larger movement in the opposite direction. If we take the initial movement, lasting from 9.6 to 9.10 P.M. G.M.T. on August 20, 1902, we find for the amplitude of the resultant disturbing force 38  $\gamma$  at Winter Quarters, 14  $\gamma$  at Kew, 11  $\gamma$  at Colaba, and 6  $\gamma$  at Mauritius and Christchurch. Thus the stations are in the same order, as regards magnitude, as on April 5, 1903. Further, the vectors were fairly similarly directed on the two occasions. Thus at Kew, the resultant vector on August 20 lay in the meridian  $214^\circ$  E., and made an angle of  $61^\circ$  with the Earth's axis, while on April 5 the corresponding angles were respectively  $224^\circ$  and  $68^\circ$ .



§ 16. Attention may be directed for a moment to a phenomenon presented by the Falmouth horizontal force curve of November 6, 1902 (not reproduced). A sudden commencement is recognisable at 3.52 P.M., but on looking closely at the initial movement it is seen to be distinctly double. There is first a small decrease, lasting only about  $1\frac{1}{2}$  minute, and the subsequent rise to a maximum at 3.58 is markedly checked after it has been in progress about  $1\frac{1}{2}$  minute. The different stations, it may be remarked, agree apparently as to the time of commencement, but not as to the time of reaching the summit.

Turning to the Antarctic curves of November 6, 1902 (not reproduced), we find a conspicuous oscillation in the horizontal force. An increase lasting about  $1\frac{1}{2}$  minute appears to synchronise with the short decrease at Falmouth; it is immediately followed by a larger increase which went on for six minutes after the summit appeared in the Falmouth curve. The other elements in the Antarctic were also disturbed. The vertical force, like the horizontal, shows a to and fro movement, but the second turning point seems slightly to precede that in the horizontal force. Owing presumably to the rapidity of the movement, and consequent faintness of the trace, details of the change of declination are partly lost.

§ 17. The next slide, fig. 5, shows a disturbance having a sudden commencement, which though comparatively small is in some ways classic. At Falmouth, Colaba, Christchurch and Mauritius (the last-mentioned curve is not reproduced) a sudden rise of horizontal force began at 11.59 A.M. G.M.T. on May 8, 1902, introducing a disturbance whose termination a few minutes after 8 P.M. is unusually well marked. There is a remarkable parallelism between several of the movements experienced at the different stations, notably from 2.10 to 2.28 P.M. and from 7.4 to 7.11 P.M. The commencement, as already remarked, is at or very close to 11.59 A.M., and the great eruption of Mont Pelée in Martinique which overwhelmed St. Pierre began at about 11.55 A.M. G.M.T. Dr. L. A. Bauer, now of the Carnegie Institution, Washington, noticed the practical agreement between the time of the Mont Pelée eruption and the commencement of the magnetic storm recorded at American observatories, and he made the suggestion that the coincidence might not be accidental, but that the magnetic movement might be a consequence of what happened at Martinique. It was clearly not a case of mere earthquake

tremors affecting the magnetographs, because the interval between the two events was too short to admit of even the rapidly travelling "preliminary tremors" reaching the observatories. Further, if it had been a direct seismic effect, it would have reached different stations at times growing later as the distance from Martinique increased. Thus what Dr. Bauer suggested the possibility of was a magnetic effect, arising presumably from displacement of magnetic matter at Mont Pelée, or from some direct influence of the eruption on the Earth's magnetism. Any suggestion of Dr. Bauer's naturally commands attention. I have thus gone into the question more fully than I should otherwise have done; but I may say at once that the conclusion I have reached is that the Mont Pelée eruption had probably no direct connection with the magnetic storm.

Before explaining the reasons for this view, I would first direct attention to one feature of the Falmouth horizontal force trace in fig. 5. It has been perhaps a little exaggerated by the Assistant who traced the curve from which the photograph was derived. What the original curve shows is rather a temporary hesitation to rise than a distinct decrease occurring at 12.3. There was, I think, a real decrease, but it was somewhat obscured in the original curve owing to the size of the spot of light which produced the photographic trace. The phenomenon is seen rather more clearly in the Kew traces, both horizontal force and declination. At Colaba there is only a slight indication of a pause in the horizontal force movement. No pause is visible at Mauritius or Christchurch, but the photographic copy available of the Christchurch curve was somewhat fuzzy, and the copy of the Mauritius curve was a tracing. Thus the absence of the phenomenon at these two stations is by no means certain.

In the Antarctic, as fig. 6 shows, we have in the declination curve a conspicuous double movement, the first peak occurring about 12.3 G.M.T. (11.10 P.M. L.T.) on May 8, *i.e.* at the time when the Falmouth curve shows a halt. The second peak in fig. 6 occurs 2 or 3 minutes later than the prominent peaks at Falmouth and Colaba; but the summits are all somewhat rounded, the main part of the movement being over by 12.8 or 12.9 G.M.T.

In the Antarctic, as elsewhere, there is more than usual definiteness both in the beginning and end of the 8-hours disturbance. It will be noticed that the movements during the

first half of the storm are considerably smaller than those during the second half, which occurred hours after St. Pierre was overwhelmed. The movements about 7 P.M. G.M.T. (6 A.M. L.T. on the 9th) were at all the stations decidedly the most rapid, so that the activity of the operating cause would seem to have been greatest about seven hours after the eruption. The rapidity of the Antarctic movements at this period led in fact to some loss of trace.

§ 18. Curiously enough, Colaba, Mauritius and Christchurch are pretty nearly equidistant from Mont Pelée, and Winter Quarters though somewhat nearer is not very much. Kew and Falmouth on the other hand are at only about half the angular distance of the first-mentioned three stations. One would expect any magnetic disturbance due to a volcanic eruption to be more or less symmetrical round the vertical at the spot where the eruption occurred, and to decrease rapidly as the distance from this spot increases. Thus if the source of the magnetic disturbance were Mont Pelée, one would expect the magnetic disturbances at Colaba, Mauritius, Christchurch and Winter Quarters to be small compared with those at Kew or Falmouth, and to be roughly equal amongst themselves. The vector, however, answering to the disturbance from 11.59 to 12.9 was only  $19.2 \gamma$  at Kew as compared with  $21.6 \gamma$  at Colaba; while its value at the latter station was much larger than the values at Mauritius and Christchurch, which were respectively only  $11.1 \gamma$  and  $9.1 \gamma$ . Again, if we calculate the vectors answering to the magnetic disturbances between 11.59 and 12.3 and between 12.3 and 12.9 separately, we obtain at Winter Quarters  $28 \gamma$  for the former and  $52 \gamma$  for the latter, as against  $9.3 \gamma$  and  $10.2 \gamma$  respectively at Kew. The commencing disturbance was in short much larger at Winter Quarters than at any of the other stations.

For the vectors answering to the movement later in the day from 7.4 to 7.11 P.M. G.M.T. we find for the amplitudes  $23.1 \gamma$  at Kew,  $19.8 \gamma$  at Colaba,  $7.0 \gamma$  at Mauritius, and  $9.8 \gamma$  at Christchurch, indicating a very similar distribution of force intensity to that of the commencing movement. The vectors moreover for the two movements were nearly parallel to one another at Kew, and still more so at Colaba.

If the commencing movement on May 8 had been due to so exceptional a cause as the Mont Pelée eruption, one would naturally have expected it to take an unusual form. But it was

of a type which showed a close resemblance not merely to the movement from 7.4 to 7.11 P.M. on the same day, but also to the commencing movements of several other storms which occurred in 1902 and 1903. Thus denoting the magnitude of the resultant vector by  $\Delta R$ , its inclination to the Earth's rotation axis by  $\theta$ , and the east longitude of the meridian plane containing it by  $\phi$ , we have the following results for commencing disturbances :—

Date of Storm.	Kew.	Colaba.	Mauritius.	Christchurch.
	$\Delta R$ $\theta$ $\phi$	$\Delta R$ $\theta$ $\phi$	$\Delta R$ $\theta$ $\phi$	$\Delta R$ $\theta$ $\phi$
May 8, 1902 ...	$\overset{\gamma}{19\cdot2}$ $\overset{\circ}{61}$ $\overset{\circ}{225}$	$\overset{\gamma}{21\cdot6}$ $\overset{\circ}{21}$ $\overset{\circ}{159}$	$\overset{\gamma}{11\cdot1}$ $\overset{\circ}{58}$ $\overset{\circ}{17}$	$\overset{\gamma}{9\cdot1}$ $\overset{\circ}{46}$ $\overset{\circ}{208}$
Aug. 20, 1902...	14.4 61 214	10.6 16 203	5.9 21 32	5.7 49 135
April 5, 1903 ...	45.0 68 224	26.8 11 167	...	22.6 40 148
Aug. 25, 1903...	34.6 56 214	17.7 11 272		

These results point to the conclusion that the causes in operation during the commencement of the magnetic storm of May 8, 1902, were not of an exceptional kind.

§ 19. I now pass to what appears at first sight a remarkable example of the recurrence of a magnetic storm after Mr. Maunder's interval of 27 days. Fig. 7 shows an Antarctic disturbance occurring about 6 P.M. (L.T.) on July 26, 1903. I would specially direct attention to the vertical force and declination traces. The sensitiveness of the horizontal force magnet was so excessive that one is apt to form an exaggerated idea of the relative importance of the disturbances in that element. They were in reality usually notably less than those in the declination. In fig. 7 the declination curve shows a nearly symmetrical movement up and down the sheet. During the downward movement in D (*i. e.* increase of the element) the vertical force curve runs up the sheet (also increase of the element). During the first movement in D the V curve appears nearly level, but indicates in fact a very small decrease in force. The to and fro movement in D occupied about an hour. The H curve describes a movement similar in character to that in D but opposite in direction. Both the D and H curves went beyond the limit of registration, but apparently not very far.

FIG. 7.—Antarctic, July 26, 1903.

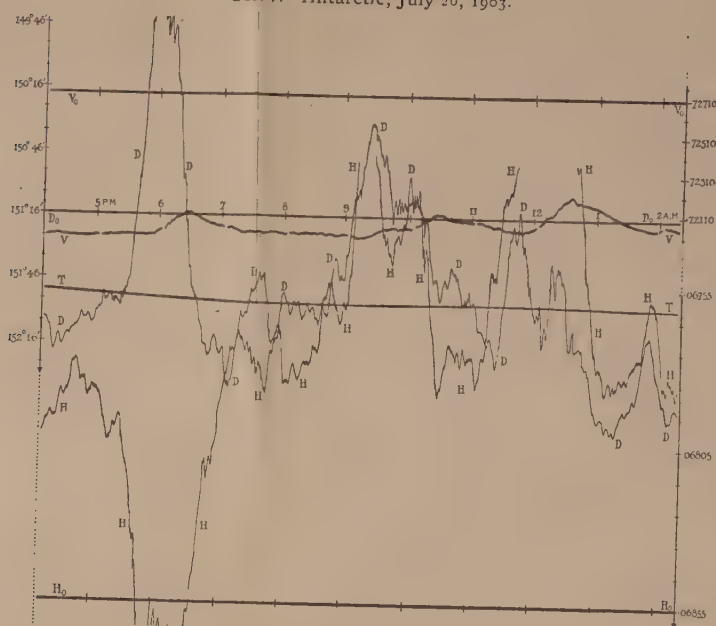


FIG. 9.—Antarctic, June 28, 1903.

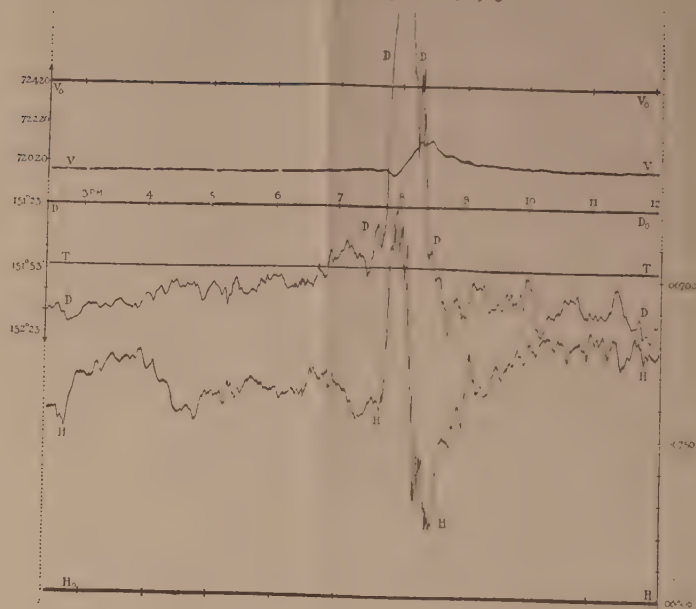


FIG. 8.—Antarctic, June 29, 1903.

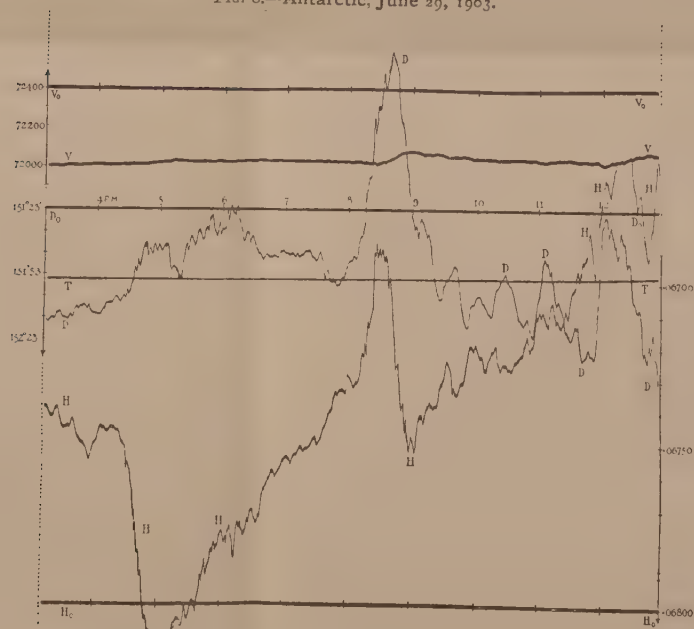
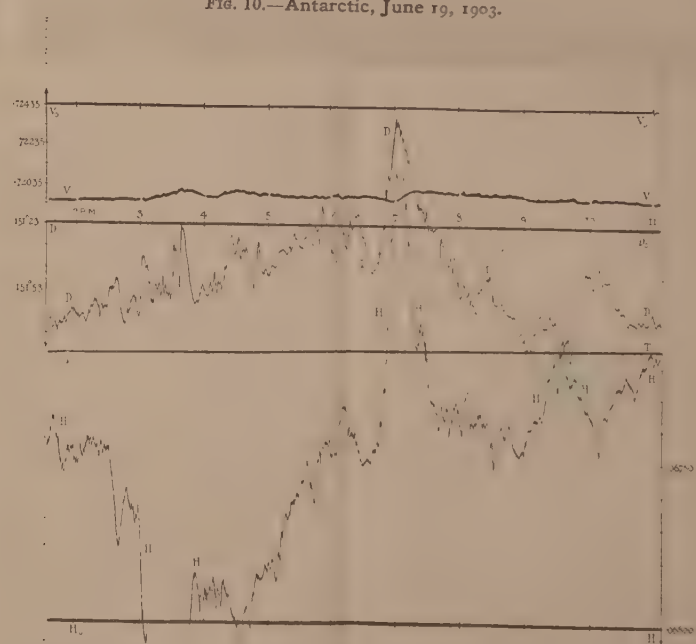


FIG. 10.—Antarctic, June 19, 1903.







The next curve, fig. 8, shows what was happening at Winter Quarters on June 29th, 1903, *i. e.* 27 days earlier. It will be observed that the D and V movements between 8 and 9 P.M. L.T. in fig. 8, and those culminating about 6 P.M. in fig. 7, present a remarkable resemblance, the chief difference being that the amplitude of movement was greater on the later occasion. The H movement, however, on June 29th was practically in phase with the D and V, instead of being in opposite phase as on July 26th. The exact interval between the two disturbances was very approximately 26 days,  $21\frac{1}{2}$  hours.

The next curve, fig. 9, shows the events of June 28th, 1903, *i. e.* the day immediately preceding that to which fig. 8 applies. The D and V movements culminating about 8 P.M. L.T. in fig. 9 present again a remarkable resemblance to those in fig. 7. We have thus a choice between a 28- and a 27-day period. Nor is our choice even thus limited, as the next curve, fig. 10, shows. It represents a disturbance culminating about 7 P.M. L.T. on June 19, or nine days earlier than the last.

§ 20. We can hardly suppose that the resemblance between these four curves means nothing more than accident. What first caught my attention was the resemblance between the curves of June 28 and 29 (figs. 9 and 8). It recalled the fact that more than one previous observer has remarked on the occasional resemblance between small declination disturbances on successive days. Resemblances of a kind are practically certain to occur in a long series of records, and I am not aware that in any previous case satisfactory evidence has been advanced that anything more than chance is *necessarily* involved. I have seen a good many instances of apparent repetitions at intervals of nearly 24 hours in the Kew curves, but none I think so striking as that afforded by figs. 8 and 9.

Perhaps the earliest occasion when the phenomenon was noticed was in a discussion of Lisbon curves by Senor Capello. This is referred to by Dr. Balfour Stewart in the '*Encyclopædia Britannica*.'\* Balfour Stewart had himself noticed analogous cases at Kew, and whilst he apparently reserved his opinion as to the real significance of the phenomenon, he suggested a possible explanation, if a true physical cause should prove to be involved. The suggested explanation—which has a surprisingly modern flavour—was to the effect

\* 9th Edition. 'Terrestrial Magnetism' (under '*Meteorology*'). Art. 87.

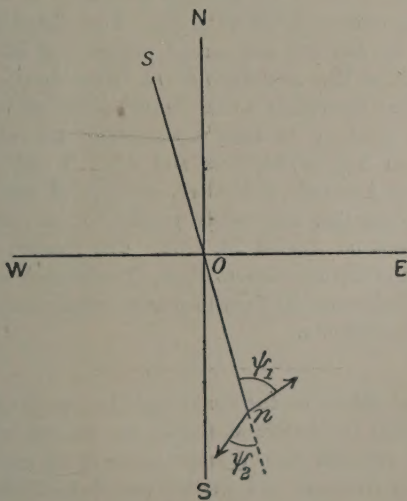
that an area near some meridian on the Earth's surface or in the superincumbent atmosphere might in some way become for several days at a time especially sensitive to solar action, so that magnetic disturbance would arise on successive presentations of that meridian to the sun.

The fact that the disturbances on June 19, 28, 29 and July 26 occurred not far from the same hour in the evening suggested to me, I must admit, for a moment an explanation of a much more prosaic character than that of Dr. Balfour Stewart; viz., that all the disturbances might have been artificial and due to misplaced activities on the part of the crew of the 'Discovery.' An instant's reflection convinced me that this could not be the true explanation, but it may occur to others. I have thus mentioned it so as to give it its *quietus*. When artificial disturbances are set up through the proximity of individuals with iron in their possession, rapid oscillations are set up in the magnets, which appear as blurred portions of trace in the records. Now the movements in figs. 7 to 10, though rapid, are in no ways blurred, but are quite sharp. Apart from these and other general considerations, I have positive proof of the genuineness of the disturbances on June 19 and July 26. These were days for which I had copies of the Christchurch curves, and on both occasions there were *prominent* disturbances there synchronising with those at Winter Quarters. Nay, further, it was possible to detect synchronous movements even at the other co-operating stations, including Kew itself.

§ 21. Before indulging in speculations as to the real nature of the phenomenon, I examined all the 600 Antarctic curves, and found no less than 82 clear examples of a disturbance showing a general resemblance to the disturbances of June 19, 28, 29 and July 26, 1903. In addition, there were a good many cases in which a lesser degree of resemblance presented itself, and others in which but for the absence of a vertical force trace one would probably have decided that a close resemblance existed. In more than half of the 82 occurrences the disturbance culminated between 7 and 9 P.M. L.T., and all except 18 took place between May and August (*i. e.* during Antarctic midwinter). The typical disturbance may be regarded as possessing two phases. During the first the three magnetic elements diminish, during the second they increase. In some individual cases, as in fig. 7, the H curve behaves oppositely to the other two, but the exceptions are comparatively few. On the average, the to and fro movements

are nearly equal in D and in H, but the increase in V during the second phase is on the average about  $3\frac{1}{2}$  times the decrease during the first phase. The average duration of the second phase slightly exceeds 20 minutes, and is about 3 minutes in excess of that of the first phase. Taking into account all the examples of the phenomenon in which there was no loss of trace, and grouping them under the months in which they occurred, I calculated for each group the mean amplitudes of the movements during the two phases. The mean derived from each group was regarded as answering to a single disturbance, and a calculation was made of the amplitude and direction of the disturbance vector to which each phase of the movement might be supposed due.

In the following table  $\psi$  denotes the inclination of the horizontal component of the vector to the magnetic meridian at Winter Quarters (about  $152\frac{1}{2}^\circ$  E), and  $\chi$  the inclination of the vector itself to the horizontal plane. The suffixes 1 and 2 relate to the first and second phases respectively, the  $\psi$  angles being measured



as shown in the figure, in which  $n$  represents the north pole of the compass-needle  $ns$ , N geographical north. A + sign attached to  $\chi$  means that the disturbing force acting on the north pole is directed upwards, the - sign denoting the contrary.

	$\psi_1$	$\chi_1$	$\psi_2$	$\chi_2$
April & May .....	71° 22'	-7° 13'	57° 22'	+21° 45'
June .....	81 25	-6 33	63 40	+26 6
July .....	77 22	-5 0	73 22	+19 46
August .....	82 32	-5 6	78 41	+18 21
September .....	73 29	-4 59	68 16	+19 35
All .....	75 53	-5 35	70 48	+20 47

The similarity of the results from the different groups points to a fairly uniform set of conditions and so presumably to a community of origin.

Examining the Kew and Christchurch curves for June 19, 28 and 29 and July 26, 1903, and the Colaba curves for the last-mentioned date, I found disturbances answering in time to the conspicuous movements in the Antarctic. I calculated the vectorial disturbances for the first and second phases, and in all cases—except June 19 at Kew and June 29 at Christchurch—it proved that the vector answering to the *second* phase showed a close approach in position to that representing the mean initial disturbance on August 20, 1902, and April 5 and August 25, 1903. In the Antarctic, somewhat curiously, it was the vector for the *first* phase that was nearly parallel to the vector for the initial movement on August 20, 1902. The relative amplitudes, however, of the disturbances at Kew, Christchurch and Winter Quarters showed a marked departure from what obtained for the sudden commencements.

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A variety of slides were then exhibited illustrating various types of disturbance in the Antarctic, and it was pointed out how the comparatively rounded wave-like movements of the evening passed into the more irregular and shorter period oscillations of the morning. These and a variety of other disturbance curves will be included in the volume now under preparation, which deals with the magnetic records obtained at the Winter Quarters of the 'Discovery' between March, 1902 and January, 1904.



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